THE CONSTRUCTION OF FINITE FACTORISABLE GROUPS.

by

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Except where otherwise stated, this thesis is the result of my own research and effort.

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ABSTRACT.

A theory of non-normal group extension is developed to construct all groups, $G = ASB$, for given groups A and B, in which each $g \in G$ has a unique expression as asb, or as b's'a', where $a,a' \in A$, $s,s' \in S$, $b,b' \in B$. A group, G, will have this property if and only if it contains A and B as fully inconjugate subgroups, i.e. for which no non-trivial subgroups of A and B are conjugate in G.

G may be expressed by a "generalised transcoset", $\{A,S,B,\chi\}$, where χ is a permutation on the elements of G (or on the formal product, ASB, in its derivation) This contains complete information about the structure of G.

For $\{A,S,B,\chi\}$ to be a generalised transcoset, χ must be chosen so that two conditions (the A- and B-commuting conditions) are satisfied. The generalised transcoset must satisfy further straightforward conditions in order to define a group over ASB. Thus the construction of groups, $G = ASB = BSA$, is a special case of the problem of constructing generalised transcosets, of interest in its own right.

A transcoset of A is an ordered triple, $\{A,R,\tau\}$, satisfying the A-commuting condition. $\{A,S,B,\chi\}$ can be interpreted as simultaneously a transcoset of A, viz. $\{A,(SB),\chi\}$, and a transcoset of B, viz. $\{B,(SA),\chi^{-1}\}$. All transcosets of A can be partitioned into, and concatenated from, elementary trans-

ABSTRACT. (continued)

cosets of A, for which the structure is determined.

The problem of constructing all generalised transcosets of A and B, i.e. transcosets of A which are also transcosets of B, is considered, both for the special case of $S = E$, yielding G as a Zappa product, AB, $A \cap B = E$, and for the case of non-trivial S, but with A and B both of order 2, for which a schematic method is introduced.

A computer investigation of the elementary transcosets of $A = D_8$, the dihedral group of order 8, is discussed, in the context of Zappa products.

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CHAPTER I. On Groups generated by two fully inconjugate Subgroups.

1.1. Theories of Group Extension.

1.1.1.1. DEFINITION. Two subgroups, A and B, of a group G are fully inconjugate in G if and only if no subgroup of A is conjugate in G to a subgroup of B.

It follows immediately that if A and B are of relatively prime order (finite) then they are fully inconjugate.

This report treats the problem of constructing all finite groups, G, which are generated by any two given finite non-trivial groups, A and B, such that A and B are fully inconjugate in G. This problem is closely related to that of embedding a group A in a group G as a subgroup. G is sometimes called an extension of A (e.g. by M. Hall, jnr. (1959)). If use is made of a second group, B, in constructing G then G may be called an extension of A by B, or for particular cases, a product of A and B.

It is instructive briefly to review previous work on the topic of group extensions of A by B, where A and B are given finite groups.

O. Schreier (1926) showed how to construct all extensions, G, of A by B where A becomes a normal subgroup of G, and B becomes the factor group G/A.

G. Zappa (1940) treated the class of permutable products of A by B. These are groups, G, in which every element of G can be expressed as ab, or as b'a', for some $a, a' \in A$, $b, b' \in B$. Writing AB for the set $\{ab \mid a \in A,\ b \in B\}$, then $G = AB = BA$. Neither A nor B need be normal in G.

Zappa imposed the restriction $A \cap B = E$, the trivial group. Under this restriction, each $g \in G$ has a unique expression in each of the forms ab or b'a'.

1.1.1.2. DEFINITION. A <u>Zappa product</u> is a group $G = AB = BA$, $A \cap B = E$. Subgroups A and B are called <u>factors</u> of G.

1.1.1.3. PROPOSITION. If G is a Zappa product of factors A and B, then A and B are fully inconjugate.

PROOF. Otherwise for some non-trivial $a' \in A$, $b' \in B$,

$$b' = g^{-1}a'g \text{ for some } g \in G.$$

Now $g = ab$ for some $a \in A$, $b \in B$.

Therefore $b' = (ab)^{-1}a'ab = b^{-1}a^{-1}a'ab$,

whence $bb'b^{-1} = a^{-1}a'a \neq 1$, contradicting the assumption of $A \cap B = E$.

Hence Zappa products fall as a special case within the scope of the present work. They will be treated further in Chapter 3.

Zappa obtained necessary and sufficient conditions upon two sets of permutations for them to define a Zappa product of the two given finite groups, A and B. The two sets of permutations are representations of B on the elements of A, and representations of A on B respectively. His conditions bear a presumably intentional resemblance to the necessary and sufficient conditions for a given automorphism group and factor set to define a Schreier extension of one given group by another.

J. Szép and L. Rédei (1950, 1951) generalised the work of Zappa by relaxing the condition $A \cap B = E$, in order to derive non-simplicity criteria for certain classes of permutable products, $G = AB$.

L. Rédei (1950) introduced a highly generalised form of extension into Group Theory called the <u>skew product</u>. The skew product of A by B, where A and B are groups, is defined as the set of ordered couples $\{[a,b] \mid a \in A,\ b \in B\}$ under a binary operation, $\circ$, defined by:

$$[a_1,b_1] \circ [a_2,b_2] = [\alpha(a_1,b_1,a_2,b_2),\ \beta(a_1,b_1,a_2,b_2)]$$

where α and β are functions mapping the domain of arguments (A,B,A,B) onto A and B respectively.

Rédei exhibited a particular case of the skew product where α and β are of a certain restricted form, for which he obtained a necessary and sufficient set of sixteen conditions for $A \circ B$ to be a group. This restricted skew product is noteworthy because the Zappa product of A and B and the Schreier extension of A by B are both special cases of $A \circ B$.

Rédei thus succeeds in generalising Schreier's extension theory to include the Zappa product, AB, as a (generalised) extension of A by B.

What might be the value of doing so? Quite apart from the conceptual elegance of a unified theory of group extension, irrespective of how inconvenient it may be in practical use, there is the attraction of finding an analogy to the composition series of a group by using the concept of the Zappa product factor instead of that of the normal subgroup.
A finite group, G, may be reassembled from the factor groups G_{i-1}/G_i of its composition series $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset E$ by the use of Schreier extension theory. If G is soluble, a fully refined composition series will yield cyclic prime order factor groups, G_{i-1}/G_i, otherwise some of these will be simple groups (of composite order).

Simple groups have only trivial composition series. However many simple groups have a non-trivial expression as a repeated Zappa product (e.g. Alt(n), provided n/2 is not an odd integer). A solvable group of composite order may be expressed as a repeated Zappa product of its co-prime order Sylow subgroups, since for every prime number, p dividing $|G|$ there exists a p-complement (itself soluble), i.e. a subgroup whose index in G is the highest power of p dividing $|G|$ (see for example M. Hall, jnr. (1959), p 141 _et seq_.).

Hence Zappa product extension theory allows not only all solvable groups to be constructed from their Sylow subgroups, but some simple groups as well.

However, not all simple groups are expressible as permutable products, let alone as Zappa products of which one factor is a Sylow subgroup. PSL(2,13), the projective special linear group of order $\frac{1}{2}p(p+1)(p-1)$, where p in this case is 13, is not factorisable, to take one example (see W.R. Scott (1964)). However, to take the same example, PSL(2,13) may be expressed as ASB, where S is a complex of 12 elements, and A,B are cyclic groups of orders 13 and 7 respectively. Each element in PSL(2,13) then has a unique expression in the form $asb = b's'a'$, for some $a,a' \in A$, $s,s' \in S$, $b,b' \in B$, provided S is a common transversal of the two sets of distinct double cosets, $\{AgB \mid g \in G\}$, $\{Bg'A \mid g' \in G\}$.

The need is thus indicated for an extension theory which can handle simple groups which are not Zappa products. The theory here presented may be viewed as a generalisation of the Zappa product; not in the obvious manner of relaxing the condition $A \cap B = E$ where A and B are the factors, but so as to express every element of the generalised extension G of A by B uniquely as either asb or as b's'a', $a \in A$, $s \in S$, $b \in B$, where S is some particular choice of complex in G. For such an S to exist it is necessary and sufficient for A and B to be fully inconjugate in G, as shown in subsection 1.2.1.

Our generalised extension theory can construct all finite groups generated by two fully inconjugate subgroups, A and

B, where A and B are given. It is an open question whether or not this includes all finite simple groups, connected with the question whether or not all finite simple groups can be generated by just two elements.

Another motive for developing the present theory is to facilitate the actual construction of extensions of A by B with particular properties (e.g. see subsection 2.2.6). This is something not immediately accessible to either Zappa theory or Schreier theory as they stand.

1.2. Groups generated by two fully inconjugate Subgroups.

Let G be a finite group and let A and B be two fully inconjugate subgroups which generate G. We shall express G by an ordered quadruple, $\{A,S,B,\chi\}$, where S is the union of $E = A\cap B$ and a set of indeterminates, and χ is a 1-1 mapping of the formal product ASB onto itself. We shall describe χ as a permutation on ASB.

It will be shown that χ, together with information about the group structure of A and B, contains complete information about the group structure of G. Moreover if there are no non-trivial subgroups of A which are normal in the group G, then separately presented information about the group structure of A is redundant, since it is already contained in χ. A similar situation holds for B.

1.2.1. Expressing group G generated by fully inconjugate subgroups, A and B, as G = ASB = BSA.

We shall establish that for all such groups G there exists a complex, $S\subset G$, such that every element, $g\in G$ has a unique expression in the form asb, or alternatively as b's'a', $a,a'\in A$, $b,b'\in B$, $s,s'\in S$. We may then express G as ASB = BSA. Note that this will not occur unless A and B are fully inconjugate.

1.2.1.1. THEOREM.(P. Hall). Any two distinct partitions of a finite set (in this case G) into the same number of equal sized parts possess a <u>common transversal</u>, S; i.e. a transversal (a system of distinct representatives of the parts) of one partition that is also a transversal of the other partition.

For a proof of this theorem see H.J. Ryser (19).

Therefore for any two fully inconjugate subgroups, say A and B, there exists a common transversal, S, of the two partitions of set G into distinct double cosets of the forms AgB and Bg'A respectively, where $g, g' \in G$.

1.2.1.2. PROPOSITION. Every element $g \in G$ may be expressed uniquely as asb, or alternatively as b's'a', where $a, a' \in A$, $b, b' \in B$, $s, s' \in S$, and A and B are fully inconjugate.

PROOF. (i) Assume $g = a_1s_1b_1 = a_2s_2b_2$, where $a_1, a_2 \in A$, $s_1, s_2 \in S$, $b_1, b_2 \in B$.

Double cosets, As_1B, As_2B are distinct, hence disjoint, if $s_1 \neq s_2$. Therefore $a_1s_1b_1 = a_2s_2b_2$ implies $s_1 = s_2$.

The double cosets, AgB, BgA, contain the same number, $|A| \times (B : B \cap g^{-1}Ag)$, of elements. (See M. Hall, jnr. (1959) p 15). Since A and B are fully inconjugate, $B \cap g^{-1}Ag = E$ for all $g \in G$. Therefore AgB and BgA each contain $|A| \times |B|$ distinct elements.

It follows that $a_1 = a_2$, and $b_1 = b_2$.

(ii) Similarly for $g = b's'a'$.

1.2.1.3. PROPOSITION. S may be chosen to contain any given element, $x \in G$.

PROOF. Let a common transversal, $\underline{S}$ be chosen not containing x. Suppose for $s, s' \in \underline{S}$, $BxA = BsA$, $AxB = As'B$.

If AsB contains x, then $AsB = AxB = As'B$, whereupon $s = s'$. s is then the representative of both BxA and AxB and may simply be replaced by x to yield a new common transversal, S.

Therefore assume that AsB does not contain x. Then $AsB \neq BsA$ whereupon there exists $s_2 \neq s$ (yielding $Bs_2A \neq BsA$) such that $Bs_2A \cap AsB$ is non-trivial.

Furthermore $As_2B \neq Bs_2A$; or else $As_2B = Bs_2A$ has a non-trivial intersection with AsB, which implies $s_2 = s$ contradicting the assumption that it does not.

Hence there must exist $s_3 \in S$, $s_3 \neq s_2$ (yielding $Bs_3A \neq Bs_2A$) such that $Bs_3A \cap As_2B$ is non-trivial.

The argument repeats until $s_i = s'$, which it will do eventually because S is finite.

Thus a sequence exists, $s=s_1, s_2, s_3, \ldots, s_i=s'$, of distinct elements of $\underline{S}$ with the following property:

$As_1B \cap Bs_2A \neq \Phi$, the empty set.

$As_2B \cap Bs_3A \neq \Phi$,

......

$As_{i-1}B \cap Bs_iA \neq \Phi$.

We may furthermore suppose there to be no shorter sequence.

Define $r_1, r_2, \ldots, r_{i-1}$ as follows:

$r_j \in As_jB \cap Bs_jA$, where $j = 1,2,\ldots,(i-1)$.

The $r_1, \ldots, r_{i-1}$ are all distinct, otherwise suppose that $r_j = r_k$, $j < k$. Then the sequence:

$As_1B \cap Bs_2A \neq \Phi$,

$As_2B \cap Bs_3A \neq \Phi$,

......

$r_j \in As_jB \cap Bs_{j+1}A \neq \Phi,$

......

$r_k \in As_kB \cap Bs_{k+1}A \neq \Phi,$

......

may be shortened by replacing the bracketed section by:

$r'_j \in As_jB \cap Bs_{k+1}A \neq \Phi,$

contradicting the assumption that no shorter sequence exists. Thus the $r_1, \dots r_{i-1}$ form a set of representatives of both $As_1B, \dots, As_{i-1}B$ and $Bs_2A, \dots, Bs_iA$.

By assumption the element x is contained in both $AxB = As_iB$ and $BxA = Bs_1A$.

Hence the set $x, r_1, r_2, \dots, r_{i-1}$, is an alternative set of distinct representatives to the set $s_1, s_2, \dots, s_i$.

Hence S may be chosen to contain any given element, x, of G.

We will have occasion to use this theorem for $x \neq 1$ later. However, we are concerned at present only with the case $x = 1$, the identity of G, We may summarise with a theorem:

1.2.1.4. THEOREM. If G is a finite group containing two fully inconjugate subgroups, A and B, then there exists $S \subseteq G$ such that

- S contains the identity, 1, of G,
- For all $g \in G$, g has a unique expression as asb, or alternatively as b's'a', $a,a' \in A$, $s,s' \in S$, $b,b' \in B$.

1.2.2. The Permutation, χ.

Given the finite group, G, two fully inconjugate subgroups, A and B, and S, a common transversal containing the two double coset expansions, $\{AgB \mid g \in G\}, \{BgA \mid g \in G\}$, we now introduce a mapping, $\chi: G \to G$, the fourth member of the ordered quadruple $\{A, S, B, \chi\}$, by which we will define group G.

1.2.2.1. DEFINITION. Let χ: asb $\to$ bsa, for all $a \in A$, $s \in S$, $b \in B$.

Since theorem 1.2.1.4. allows us to assume $S \ni 1$, (which we shall do) then:

1.2.2.2. PROPOSITION. χ is a permutation on the elements of G fixing A, S, and B.

PROOF. χ: 11b $\to$ b11, or χ: b $\to$ b,

χ: 1s1 $\to$ 1s1, or χ: s $\to$ s,

χ: a11 $\to$ 11a, or χ: a $\to$ a,

for all $a \in A$, $s \in S$, $b \in B$.

Because of G = ASB = BSA, and the property of uniqueness of both expressions asb and b's'a' for $g \in G$, χ is a 1-1 mapping of G onto G, hence it is a permutation.

The significance of χ is as follows. Even if the structure of G is otherwise unknown, still each element asb $\in$ ASB, the formal product of sets A, S and B, corresponds uniquely to some element in G. We agree to identify the formal product ASB with the conventional group product of complexes, ASB in G. This will allow us to define a unique abstract group, G, by means of the ordered quadruple, $\{A, S, B, \chi\}$, as we shall show in the next section.

1.2.2.3. DEFINITION. An ordered quadruple, $\{A, S, B, \chi\}$, is said to <u>define a group</u> if and only if there exists a group G generated by subgroups A and B, and S as a complex of G containing 1, such that

- for all $g \in G$, g has a unique expression as asb or alternatively as b's'a',
- χ maps asb onto bsa,

for all $a, a' \in A$, $s, s' \in S$, $b, b' \in B$.

We next show that the group, G, so defined is abstractly unique.

1.2.3. <u>Obtaining the left (or right) regular representation of G from the ordered quadruple, $\{A, S, B, \chi\}$.</u>

Form the formal product, ASB, from the sets A, S and B. Knowing the structure of groups A and B, the left regular representation of G, namely λ, maps A onto $\lambda(A)$, where for $a' \in A$

1.2.3.1. $$\lambda(a'): asb \longrightarrow a'(asb) = (a'a)sb \in ASB, \text{ for all } asb \in ASB.$$

λ also maps B onto $\lambda(B)$, where for $b' \in B$

1.2.3.2. $$\lambda(b'): bsa \longrightarrow b'(bsa) = (b'b)sa \in BSA = ASB, \text{ for all } bsa \in BSA = ASB.$$

We thus obtain $\lambda(A)$ as a set of permutations acting upon the set ASB, and $\lambda(B)$ as a set of permutations also acting upon ASB, in a less straightforward way.

Since ~~If~~ groups A and B generate G, then $\lambda(A)$ and $\lambda(B)$ generate $\lambda(G)$.

The above argument may be pursued in left/right dual form to treat the right regular representation, ρ, of G.

1.2.4. Redundancy of information about the structure of A and B.

In the previous section we showed how to obtain a regular representation of G from an ordered quadruple, $\{A, S, B, \chi\}$, which defines it. It is implied that A and B are both sets over which given groups are defined, but that no internal multiplicative structure is given for S (apart from the element 1).

In this section we show how to obtain a representation of A, and similarly of B, from $\{A, S, B, \chi\}$, without knowing the group structure of either A or B. Such a representation is, however, not necessarily faithful.

1.2.4.1. DEFINITION. Let $\nu(A)$ be the representation of A on the right cosets of A in G. Thus, for given $a' \in A$,

$$\nu(a'): Ag \longrightarrow A(ga'), \text{ all } g \in G.$$

$\nu(a')$ thus permutes right cosets of A in G amongst themselves in a manner that can be discovered in terms solely of the set ASB and the permutation χ as follows:

Given $a' \in A$, for all $b' \in B$, $s' \in S$,

$$b's'a' = (a's'b')\chi = asb, \text{ say, for some } a \in A,\ s \in S,\ b \in B.$$

Thus $\nu(a'): A(b's') \longrightarrow (Ab's')a' = A(asb) = A(sb)$,

or $\nu(a'): A((s'b')\chi) \longrightarrow A(sb)$.

1.2.4.2. PROPOSITION. (i) If G contains no subgroup of A as a normal subgroup (besides E), then it is superfluous to give the group structure of A in the ordered quadruple, $\{A, S, B, \chi\}$.

(ii) Similarly for B.

PROOF. The representation, ν, of A on the right cosets of A in its supergroup G, is faithful if and only if no non-trivial subgroup of A is normal in G.
ν, and hence the group structure of A, may be obtained from χ, having merely been provided with A, S, B as structureless sets intersecting in one common element, called 1.

It follows that if no non-trivial subgroups of A or B are normal in G, and in particular, if G is simple, then χ contains complete information about the structure of G. Thus we can completely describe the structure of any simple group, G, which can be generated by two non-trivial, fully inconjugate subgroups, A and B, by a single permutation of degree $|G|$, over the formal product, set ASB, fixing $|A| + |S| + |B| - 2$ elements.

1.3. For what choice of χ does $\{A, S, B, \chi\}$ define a group?

In section 1.2. we showed that a finite group, G, generated by two fully inconjugate subgroups, A and B, can be defined by an ordered quadruple, $\{A, S, B, \chi\}$, where S is the union of $E = A \cap B$ and a set of indeterminates, and χ is a permutation upon the set ASB, fixing $A11 \cup 1S1 \cup 11B$.

We now suppose that we are given two finite groups, A and B, a set S and a permutation χ, all with the above properties. For what choice of χ does the ordered quadruple, $\{A, S, B, \chi\}$ define a group over ASB, in the sense of definition 1.2.2.3.?

We shall obtain a set of necessary and sufficient conditions on χ for this to happen. These conditions have been chosen with the following considerations in mind:

- Ease of checking whether satisfied by given χ ,
- Ease of interpreting χ in the resulting group, G,
- Use of the conditions to construct examples of G.

They are independent, as we shall show by relaxing each condition in turn and exhibiting ordered quadruples satisfying the remaining conditions, yet not defining groups.

1.3.1. Necessary conditions on $\{A, S, B, \chi\}$.

Suppose an ordered quadruple, $\{A, S, B, \chi\}$, to be given as in 1.3.

1.3.1.1. DEFINITION. $\lambda(A)$, $\lambda(B)$, $\rho(A)$, $\rho(B)$ are the <u>derived groups</u> of $\{A, S, B, \chi\}$, where, for each $a' \in$ group A, $b' \in$ group B,

$$\lambda(a'): asb \longrightarrow (a'a)sb,$$

$$\rho(a'): (asb)\chi \longrightarrow ((aa')sb)\chi ,$$

$$\lambda(b'): (asb)\chi \longrightarrow (as(b'b))\chi ,$$

$$\rho(b'): asb \longrightarrow as(bb').$$

They are all permutation groups defined over the formal product, set ASB.

We avoid mentioning the set BSA at this stage, in order to achieve uniformity in the definition of all permutations to be introduced. These will in general be defined over the formal product ASB.

The notation is intended to be compatible with that of subsection 1.2.3. However, to assist comprehension, note that if $(asb)\chi$ is read as bsa, then the definition of $\lambda(b')$, for example, corresponds to 1.2.3.2.

1.3.1.2. PROPOSITION. For all $a' \in A$, $b' \in B$, $\lambda(a')$ commutes with $\rho(b')$, and $\lambda(b')$ commutes with $\rho(a')$.

PROOF. By definition 1.3.1.1., for all $asb \in ASB$,

$$(asb)\lambda(a')\rho(b') = ((a'a)sb)\rho(b') = (a'a)s(b\ b');$$

$$(asb)\rho(b')\lambda(a') = (as(bb'))\lambda(a') = (a'a)s(b\ b');$$

Hence $\rho(b')\lambda(a') = \lambda(a')\rho(b')$.

Likewise

$$(asb)\chi\rho(a')\lambda(b') = ((aa')sb)\chi\lambda(b') = ((aa')s(b'b))\chi;$$

$$(asb)\chi\lambda(b')\rho(a') = (as(b'b))\chi\rho(a') = ((aa')s(b'b))\chi.$$

Hence $\rho(a')\lambda(b') = \lambda(b')\rho(a')$.

1.3.1.3. PROPOSITION. If $\{A, S, B, \chi\}$ defines a group over set ASB then the following conditions must be obeyed:

1.3.1.4. (i) For no $S' \subset S$ does χ map $AS'B$ onto itself.

(ii) For all $a, a' \in A$, $\lambda(a)$ commutes with $\rho(a')$.

(iii) For all $b, b' \in B$, $\lambda(b)$ " " $\rho(b')$.

(iv) χ fixes $A11 \cup 11B \subseteq ASB$.

(v) χ fixes $1S1 \subseteq ASB$.

NOTE. We shall refer to conditions (ii) and (iii) as the <u>A-commuting</u> and <u>B-commuting conditions</u> respectively.

PROOF. If for some $S' \subset S$, $(AS'B)\chi = AS'B$, then from definition 1.3.1.1. $gp\{\lambda(A), \lambda(B)\}$ is transitive on $AS'B$ or some subset. Therefore it is intransitive on ASB, whereupon A and B cannot generate the whole of G = ASB.

The right regular representation of a group centralises the left regular representation, i.e. every member of one commutes with every member of the other. The A- and B-commuting conditions are just special cases of this general property.

Lastly, in order for $\{A, S, B, \chi\}$ to define a group, G = ASB, then for all $a \in A, s \in S, b \in B$, $(asb)\chi = bsa \in G$, (identifying group and formal products. Since A, S and B all contain 1,

$(a11)\chi = 11a = a11$,

$(1s1)\chi = 1s1$,

$(11b)\chi = b11 = 11b$.

1.3.2. Sufficiency of the conditions on $\{A, S, B, \chi\}$.

Conditions 1.3.1.4. (i)-(v) have been shown necessary for $\{A, S, B, \chi\}$ to define a group, G, over ASB. We shall now show them to be sufficient.

The problem of doing this lies in the possibility, as yet undiscounted, that for certain choices of χ the permutation group generated by the derived groups, $\lambda(A)$, $\lambda(B)$, for arbitrary χ may not contain $|ASB|$ members, nor may it even be regular: hence unable to do duty as a left regular representation of G.

We shall not approach the problem directly through the entire group, $gp\{\lambda(A), \lambda(B)\}$ itself. Instead we shall use $\lambda(A)$ and $\lambda(B)$ to define a groupoid, $\{G, \circ\}$, over a subset, G, of ASB, and then show that the conditions, 1.3.1.4. (ii), (iii) and (iv) imply that the groupoid is a group generated by A and B. Furthermore if the conditions, 1.3.1.4. (i) and (v) are additionally satisfied, the subset, G, becomes the whole of ASB, whereupon $\{G, \circ\}$ becomes the group, G, <u>defined by</u> $\{A, S, B, \chi\}$, in the sense of 1.2.2.3.

For simplicity we shall write the elements, a11, 1s1, 11b, $111 \in ASB$ as: a, s, b, $1 \in ASB$, respectively.

1.3.2.1. DEFINITION. Let set G be the orbit of the element, 1, in set ASB under $gp\{\lambda(A), \lambda(B)\}$.

In order to define a composition, $g \circ g' = g'\lambda(g)$ between each pair of elements, $g, g' \in G$, we must define $\lambda(g)$ uniquely. We want $\lambda(g)$ to lie in $gp\{\lambda(A), \lambda(B)\}$ and to be compatible with $\lambda(A)$ and $\lambda(B)$ themselves. Now for given $g \in G$ there may be more than one string, $a_1, b_1, a_2, b_2, \ldots, a_r, b_r$, of elements in A and B alternately, such that:

$$\lambda(a_1)\lambda(b_1)\lambda(a_2)\lambda(b_2) \ldots\ldots \lambda(a_r)\lambda(b_r) \in gp\{\lambda(A), \lambda(B)\}$$

maps 1 onto g. (There is at least one, by definition 1.3.2.1.). For each $g \in G$, one such string is selected arbitrarily and called the distinguished sequence of g.

1.3.2.2. DEFINITION. For all $g \in G$, let $\lambda(g) = \begin{cases} \lambda(a) \in \lambda(A) \text{ if } g = a \in A; \\ \lambda(b) \in \lambda(B) \text{ if } g = b \in B; \\ \lambda(a_1)\lambda(b_1)\lambda(a_2)\lambda(b_2) \ldots\ldots \end{cases}$

$\ldots\ldots\lambda(a_r)\lambda(b_r)$ otherwise, where $a_1, b_1, a_2, b_2, \ldots, a_r, b_r$ is the distinguished sequence of g.

1.3.2.3. DEFINITION. For all $g \in G$, let $\rho(g) = \begin{cases} \rho(a) \in \rho(A) \text{ if } g = a \in A; \\ \rho(b) \in \rho(B) \text{ if } g = b \in B; \\ \rho(b_r)\rho(a_r)\rho(b_{r-1}) \ldots\ldots \end{cases}$

$\ldots\ldots\rho(b_1)\rho(a_1)$ otherwise, where the sequence of arguments is the distinguished sequence of g in reverse order.

1.3.2.4. DEFINITION. Let $\{G, \circ\}$ be a groupoid over set G, with the binary composition: $g \circ g' = g'\lambda(g)$, for all $g, g' \in G$.

We may now establish that $\{G, \circ\}$ is a group, as follows.

1.3.2.5. LEMMA. The A- and B-commuting conditions on $\{A, S, B, \chi\}$ imply that for all $g, g' \in G$, $\lambda(g)$ commutes with $\rho(g')$.

PROOF. For all $a, a' \in A$, $b, b' \in B$,

$\lambda(a)$ commutes with $\rho(a')$ (A-commuting condition)

$\lambda(a)$ " " $\rho(b')$ (1.3.1.2.)

therefore $\lambda(a)$ " " $\rho(g')$, for all $g' \in G$.

Similarly $\lambda(b)$ " " $\rho(b')$ (B-commuting condition)

$\lambda(b)$ " " $\rho(a')$ (1.3.1.2.)

therefore $\lambda(b)$ " " $\rho(g')$, for all $g' \in G$.

Hence $\lambda(g)$ " " $\rho(g')$, for all $g, g' \in G$.

1.3.2.6. LEMMA. For all $a \in A$, $b \in B$, $g \in G$,

$$g\rho(b) = b\lambda(g),$$
$$g\rho(a) = a\lambda(g).$$

PROOF. By definition 1.3.1.1., $\rho(b): 1 \rightarrow b$.

Also, since χ fixes A, $\rho(a): (111)\chi = 1 \rightarrow (a11)\chi = a$.

Therefore $(\rho(b))^{-1} : b \rightarrow 1$, and lies in $\rho(B)$,

$(\rho(a))^{-1} : a \rightarrow 1$, and lies in $\rho(A)$.

By definition 1.3.2.2., $g = 1\lambda(g)$

$$= b(\rho(b))^{-1}\lambda(g)$$

$$= b\lambda(g)(\rho(b))^{-1} \quad \text{by lemma 1.3.2.5.}$$

Multiplying both sides by $\rho(b)$:

$$g\rho(b) = b\lambda(g)(\rho(b))^{-1}\rho(b) = b\lambda(g).$$

In a similar fashion,

$$g = 1\lambda(g) = a(\rho(a))^{-1}\lambda(g) = a\lambda(g)(\rho(a))^{-1},$$

whence $g\rho(a) = a\lambda(g)$.

1.3.2.7. LEMMA. For all $g \in G$, $1\rho(g) = g$.

PROOF. If $g \in A \cup B$, the lemma follows from definition 1.3.1.1.

More generally, expressing $\rho(g)$ by the distinguished sequence for g, as in definition 1.3.2.3.:

$$
\begin{aligned}
1\rho(g) &= 1\rho(b_r)\rho(a_r)\ \ldots\ldots\ldots\ \rho(b_1)\rho(a_1) \\
&= b_r\ \rho(a_r)\rho(b_{r-1})\rho(a_{r-1})\ \ldots\ \rho(b_1)\rho(a_1), \text{ by 1.3.1.1.} \\
&= a_r\ \lambda(b_r)\rho(b_{r-1})\rho(a_{r-1})\ \ldots\ \rho(b_1)\rho(a_1), \text{ by 1.3.2.6.} \\
&= a_r\rho(b_{r-1})\rho(a_{r-1})\ \ldots\ \rho(b_1)\rho(a_1)\lambda(b_r), \text{ by 1.3.2.5.} \\
&= b_{r-1}\lambda(a_r)\rho(a_{r-1})\ \ldots\ \rho(b_1)\rho(a_1)\lambda(b_r), \text{ by 1.3.2.6.} \\
&= b_{r-1}\rho(a_{r-1})\ \ldots\ \rho(b_1)\rho(a_1)\lambda(a_r)\lambda(b_r), \text{ by 1.3.2.5.}
\end{aligned}
$$

....

(in this fashion the sequence may be reversed to:)

$$
\begin{aligned}
&= a_1\lambda(b_1)\lambda(a_2)\lambda(b_2)\ \ldots\ \lambda(a_r)\lambda(b_r), \\
&= 1\lambda(a_1)\lambda(b_1)\lambda(a_2)\lambda(b_2)\ \ldots\ \lambda(a_r)\lambda(b_r), \\
&= 1\lambda(g), \text{ by definition 1.3.2.2.}
\end{aligned}
$$

Hence $1\rho(g) = 1\lambda(g) = g$, by definition 1.3.2.2.

1.3.2.8. LEMMA. For all $g, g' \in G$, $g'\rho(g) = g\lambda(g')$.

PROOF.
$$
\begin{aligned}
g'\rho(g) &= 1\lambda(g')\rho(g), \text{ by 1.3.2.2.} \\
&= 1\rho(g)\lambda(g'), \text{ by 1.3.2.5.} \\
&= g\lambda(g'), \text{ by 1.3.2.7.}
\end{aligned}
$$

We can now prove our main lemma:

1.3.2.9. LEMMA. $\{G, \circ\}$ is a group.

PROOF. (i) <u>G is closed under ($\circ$),</u>
because G is the orbit of element 1 under - and hence a transitivity set of - $gp\{\lambda(A), \lambda(B)\}$, which contains $\lambda(g)$ for all $g \in G$.

(ii) The composition, ($\circ$), is associative.

For all $g_1, g_2, g_3 \in G$,

1.3.2.10. $g_2 \circ g_3 = g_3\lambda(g_2) = g_2\rho(g_3)$, by lemma 1.3.2.8.

Therefore $g_1 \circ (g_2 \circ g_3) = g_1 \circ (g_2\rho(g_3))$, by 1.3.2.10.

$= (g_2\rho(g_3))\lambda(g_1)$, by 1.3.2.10.

$= (g_2\lambda(g_1))\rho(g_3)$, by 1.3.2.5.

$= (g_1 \circ g_2)\rho(g_3)$, by 1.3.2.10.

$= (g_1 \circ g_2) \circ g_3$, by 1.3.2.10.

(iii) There exists a right identity in $\{G, \circ\}$, (i.e. 1)

because $g \circ 1 = 1\lambda(g) = g$, by 1.3.2.2. for all $g \in G$.

(iv) There exists a right inverse, g', to every $g \in G$.

Since $\lambda(g)$ is a permutation, for all $g \in G$, some unique element of G, say g', must be mapped onto $1 \in G$ by $\lambda(g)$. Thus $g \circ g' = g'\lambda(g) = 1$.

(i), (ii), (iii) and (iv) are sufficient to prove that $\{G, \circ\}$ is a group (see, for example, M. Hall, jnr (1959) p 4)

We may state our findings so far in the form of a proposition:

1.3.2.11. PROPOSITION. Given the ordered quadruple, $\{A, S, B, \chi\}$, where:

- A and B are finite groups such that $A \cap B = E$;
- S is the union of E and a set of indeterminates;
- χ is a permutation acting on the elements of the formal product ASB;

If χ fixes $A11 \cup 11B$, and the A- and B- commuting conditions are satisfied, then there exists a group defined over the orbit, G, of 111 = $1 \in ASB$ under gp $\{\lambda(A), \lambda(B)\}$, for which the latter, restricted to G, forms the left regular representation.

We have been able to define a group over a subset, G, of ASB, for which the derived groups, $\lambda(A)$, $\lambda(B)$, restricted to G, generate the left regular representation, by assuming only some of the conditions 1.3.1.4., namely the A- and B-commuting conditions, and the condition that χ fixes $A11 \cup 11B \subseteq ASB$. The remaining conditions are:

1.3.1.4. (i) For no $S' \subset S$ does $(AS'B)\chi = AS'B$.

(v) χ fixes $1S1 \subseteq ASB$.

One might conjecture that these two conditions standing alone imply that $\{G,\circ\}$ is the whole of ASB, i.e. that the gp $\{\lambda(A), \lambda(B)\}$ is transitive on ASB. However this is not so. A more limited proposition is true:

1.3.2.13. PROPOSITION. Conditions 1.3.1.4. (i) and (v) together imply that gp $\{\lambda(A), \lambda(B), \rho(A), \rho(B)\}$ is transitive on ASB.

PROOF by induction on the order of $S' \subseteq S$. Assume $S' \ni 1$.
Let H be the orbit of 111 = $1 \in ASB$ under gp $\{\lambda(A), \lambda(B), \rho(A), \rho(B)\}$.
To prove: $H \supseteq AS'B$ for all $S' \subseteq S$.
True for $|S'| = 1$, because $H \supseteq 1\rho(B)\lambda(A) = 1B\lambda(A) = A1B$.
Suppose true for $|S'| = r < |S|$.
Then for some $s' \in S'$, $a, a' \in A$, $b, b' \in B$, $(a's'b')\chi = asb$, where s lies in S but not in S', by assumption of 1.3.1.4. (i).

Thus $H \geq s'\rho(a')\lambda(b')\lambda(A)\rho(B) = (s')\chi\rho(a')\lambda(b')\lambda(A)\rho(B)$

by 1.3.1.4.(v),

$= (a's'b')\chi\ \lambda(A)\rho(B)$

$= (asb)\lambda(A)\rho(B)$

$= AsB$, for $S \ni s \notin S'$.

Thus $H \geq AS'B \cup AsB = A(S' \cup \{s\})B$, proving the proposition for r+1.

Note that conditions (i) and (v) of 1.3.1.4. do not, in the absence of the other conditions, imply that $\mathrm{gp}\langle\{\lambda(A),\lambda(B)\}$ is transitive on ASB. We shall exhibit an example (1.3.3.4.) to demonstrate this; one in fact which obeys all except condition 1.3.1.4.(iv), i.e. that χ fixes $A11 \cup 11B \subseteq ASB$, and for which $\mathrm{gp}\langle\{\lambda(A),\ \lambda(B)\}$ is <u>not</u> transitive on ASB.

However, if $\{A,S,B,\chi\}$ satisfies all the conditions, 1.3.1.4. (i)-(v) then we do in fact have G = ASB, where G is defined by 1.3.2.1. We prove this as a corollary of the previous proposition.

1.3.2.14. COROLLARY. If $\mathrm{gp}\langle\{A,\ S,\ B,\chi\}$ satisfies all the conditions 1.3.1.4. (i)-(v) then G = ASB.

PROOF. By proposition 1.3.2.11, a group is defined over G for which $\lambda(A)$, $\lambda(B)$, restricted to G, generate the left regular representation, and $\rho(A)$, $\rho(B)$, restricted to G generate the right regular representation.
Therefore the orbit of $111 = 1$ under $\mathrm{gp}\langle\{\lambda(A),\lambda(B),\rho(A),\rho(B)\}$ is also G.

By proposition 1.3.2.13., $\{\lambda(A), \lambda(B), \rho(A), \rho(B)\}$ is transitive on the whole of ASB. Therefore G = ASB.

We may summarise the results so far obtained in section 1.3. as a theorem.

1.3.2.15. THEOREM. The set of conditions 1.3.1.4. (i)-(v) is necessary and sufficient for $\{A, S, B, \chi\}$ (as defined in 1.3.2.11) to define a group over the formal product ASB in the sense of 1.2.2.3.

PROOF. The necessity of the conditions has been shown in subsection 1.3.1.

We show sufficiency as follows:

By lemma 1.3.2.9., $\{G, \circ\}$, as defined in 1.3.2.4., is a group.

By corollary 1.3.2.14., G = ASB.

It remains only to show that, for all $a \in A$, $s \in S$, $b \in B$,

$$\{G, \circ\} \ni a_\circ s_\circ b = asb \in ASB;$$

$$\{G, \circ\} \ni b_\circ s_\circ a = (asb)\chi \in ASB.$$

Now $s_\circ b = b\lambda(s) = s\rho(b)$ (1.3.2.6.).

Therefore $a_\circ(s_\circ b) = s\rho(b)\lambda(a) = asb$, by definition 1.3.1.1.

Similarly $s_\circ a = a\lambda(s) = s\rho(a)$ (1.3.2.6.).

Therefore $b_\circ(s_\circ a) = s\rho(a)\lambda(b) = (s\chi)\rho(a)\lambda(b)$, since $s\chi = s$,

$= (asb)\chi$, by definition 1.3.1.1.

Therefore, for all $a \in A$, $s \in S$, $b \in B$,

$$a_\circ s_\circ b = asb; \quad b_\circ s_\circ a = (asb)\chi, \text{ as required by 1.2.2.3.}$$

Therefore $\{A, S, B, \chi\}$ defines a group over G = ASB.

1.3.2.16. COROLLARY. $\{B, S, A, \chi^{-1}\}$ defines the same group, G, as does $\{A, S, B, \chi\}$, provided we identify the formal product bsa with $(asb)\chi$, for all $a \in A$, $s \in S$, $b \in B$.

PROOF. $\{B, S, A, \chi^{-1}\}$ has derived groups (say) $\bar{\lambda}(A)$, $\bar{\lambda}(B)$, $\bar{\rho}(A)$, $\bar{\rho}(B)$, defined on BSA, such that for all $b, b' \in B$, $s \in S$, $a, a' \in A$,

$$(bsa)\bar{\lambda}(b') = (b'b)sa;$$

$$(bsa)\bar{\rho}(a') = bs(aa');$$

$$(bsa)\chi^{-1}\bar{\lambda}(a') = (bs(a'a))\chi^{-1};$$

$$(bsa)\chi^{-1}\bar{\rho}(b') = ((bb')sa)\chi^{-1};$$ by analogy with 1.3.1.1.

Since $bsa \leftrightarrow (asb)\chi$, then $(bsa)\chi^{-1} \leftrightarrow asb$.

Thus $(bsa)\bar{\lambda}(b') \leftrightarrow (as(b'b))\chi = (asb)\chi\lambda(b')$, showing that $\bar{\lambda}(B)$ and $\lambda(B)$ are identical under the identification $(\leftrightarrow)$.

Similarly we may show that $\bar{\lambda}(A)$ and $\lambda(A)$; $\bar{\rho}(A)$ and $\rho(A)$; $\bar{\rho}(B)$ and $\rho(B)$ are all identical under $(\leftrightarrow)$.

Therefore, interpreting $(\leftrightarrow)$ as signifying equality in group G (= ASB = BSA), then $\{B, S, A, \chi^{-1}\}$ and $\{A, S, B, \chi\}$ both define the same group, G.

1.3.3. Independence of the conditions on $\{A, S, B, \chi\}$.

We show the independence of the conditions 1.3.1.4. (i)-(v) by exhibiting examples of ordered quadruples, $\{A, S, B, \chi\}$, for which each condition is relaxed as stated, but the other conditions are obeyed.

For convenience we shall use consecutive numerals to represent elements of the set ASB. ASB and $(ASB)\chi$ will be tabulated, so that for each $a \in A$, $s \in S$, $b \in B$ the corresponding asb, $(asb)\chi$ are located in a natural fashion. A cubic tabulation is appropriate, which we represent by laying the panels AsB, etc. side by side, thus:

$$\left|\begin{array}{lcl} 1 & \cdots\cdots & b\ \cdots \\ \vdots & & \vdots \\ a & \cdots\cdots & ab\ \cdots \\ \vdots & & \vdots \end{array}\right| \begin{array}{c} \cdots\cdots\cdots \\ \\ \cdots\cdots\cdots \\ \\ \end{array} \quad \left|\begin{array}{lcl} s & \cdots\cdots & sb\ \cdots \\ \vdots & & \vdots \\ as & \cdots\cdots & asb\ \cdots \\ \vdots & & \vdots \end{array}\right|$$

$(ASB)\chi$ is tabulated as the image of the tabulated ASB under χ, thus $(asb)\chi$ will appear in the location corresponding to asb. The numeral 1 will stand for the sole element in the set $E = A \cap S \cap B$.

1.3.3.1. EXAMPLE. Relaxing condition 1.3.1.4. (i) only.

$$\left|\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}\right|\begin{array}{cc} 5 & 7 \\ 6 & 8 \end{array}\left|\begin{array}{cc} 9 & 11 \\ 10 & 12 \end{array}\right| \xrightarrow{\chi} \left|\begin{array}{cc} 1 & 3 \\ 2 & 6 \end{array}\right|\begin{array}{cc} 5 & 7 \\ 4 & 8 \end{array}\left|\begin{array}{cc} 9 & 11 \\ 10 & 12 \end{array}\right|$$

$A = \{1,2\} \cong C_2$, Fixed by χ, therefore condition (iv) obeyed.
$B = \{1,3\} \cong C_2$, " " " (iv) "
$S = \{1,5,9\}$ " " " (v) "

Derived groups: $\lambda(A) = \{(1),\ (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)\}$

$\rho(A) = \{(1),\ (1,2)(3,6)(5,4)(7,8)(9,10)(11,12)\}$

Form: $\rho^{-1}(2)\lambda(2)\rho(2) = (2,1)(6,5)(4,3)(8,7)(10,9)(12,11) = \lambda(2)$.

Therefore $\lambda(A)$, $\rho(A)$ centralise each other, hence condition (ii) is obeyed.

Derived groups: $\lambda(B) = \{(1),\ (1,3)(2,6)(5,7)(4,8)(9,11)(10,12)\}$

$\rho(B) = \{(1),\ (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)\}$

Form: $\rho^{-1}(3)\lambda(3)\rho(3) = (3,1)(4,8)(7,5)(2,6)(11,9)(12,10) = \lambda(3)$.

Therefore $\lambda(B)$, $\rho(B)$ centralise each other, hence condition (iii) is obeyed.

Condition (i) is disobeyed, since $AS'B = (AS'B)\chi$, for $S' = \{1,5\}$. As it happens, $\{A,S',B,\chi\}$, where χ is restricted to $AS'B$ does define a group (dihedral, order 8) over $AS'B$. Hence $\{A,S,B,\chi\}$ cannot possibly correspond to a group over ASB (order 12) as it would contain a group of order 8 as a subgroup.

1.3.3.2. EXAMPLE. Relaxing condition 1.3.1.4. (ii) only. Since condition (ii) is symmetrical to (iii), this serves also to exemplify relaxing condition (iii) only.

$$\left|\begin{array}{cc|cc} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{array}\right| \xrightarrow{\chi} \left|\begin{array}{cc|cc} 1 & 4 & 7 & 5 \\ 2 & 8 & 11 & 9 \\ 3 & 12 & 6 & 10 \end{array}\right|$$

By inspection condition (i) is seen to be obeyed.

$A = \{1,2,3\} \cong C_3$; fixed by χ, hence condition (iv) obeyed.

$B = \{1,4\} \cong C_2$; "

$S = \{1,7\}$; " " " (v) " .

Derived groups:

$\lambda(A) = \{(1),\ (1,2,3)(4,5,6)(7,8,9)(10,11,12) = \lambda(2),$
$\lambda(3) = \lambda^{-1}(2)\}.$

$\rho(A) = \{(1),\ (1,2,3)(4,8,12)(7,11,6)(5,9,10) = \rho(2),$
$\rho(3) = \rho^{-1}(2)\}.$

Form: $\rho^{-1}(2)\lambda(2)\rho(2) = (2,3,1)(8,9,7)(11,12,10)(5,6,4) = \lambda(2).$

Since $\lambda(2)$, $\rho(2)$ commute, then so do $\lambda(2)$, $\rho^{-1}(2)$, $\lambda^{-1}(2), \rho^{-1}(2)$, and $\lambda^{-1}(2)$, $\rho(2)$. Hence $\lambda(A), \rho(A)$ centralise each other, so condition (ii) is obeyed.

However, for the derived groups, $\lambda(B)$ and $\rho(B)$, condition (iii) is disobeyed. Thus:

$\lambda(B) = \{(1),\ (1,4)(2,8)(3,12)(7,5)(11,9)(6,10)\};$

$\rho(B) = \{(1),\ (1,4)(2,5)(3,6)(7,10)(8,11)(9,12)\}.$

Form: $\rho^{-1}(4)\lambda(4)\rho(4) = (4,1)(5,11)(6,9)(10,2)(8,12)(3,7) \neq \lambda(4).$

Thus $\rho(4)$ and $\lambda(4)$ do not commute.

$\{A,S,B,\chi\}$ fails to define a group.

1.3.3.3. EXAMPLE. Relaxing both conditions (ii) and (iii) of 1.3.1.4. This is of passing interest as the smallest example known, obeying (i), (iv) and (v) only.

$$\left|\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{array}\right| \xrightarrow{\chi} \left|\begin{array}{cc|cc} 1 & 3 & 5 & 4 \\ 2 & 6 & 8 & 7 \end{array}\right|$$ Condition (i) obeyed.

$A = \{1,2\} \cong C_2$; fixed by χ, hence condition (iv) obeyed.
$B = \{1,3\} \cong C_2$; " " " (iv) " .
$S = \{1,5\}$; " " " (v) " .

Derived groups:

$\lambda(A) = \{(1),\ (1,2)(3,4)(5,6)(7,8)\};$

$\rho(A) = \{(1),\ (1,2)(3,6)(5,8)(4,7)\}.$

Form: $\rho^{-1}(2)\lambda(2)\rho(2) = (2,1)(6,7)(8,3)(4,5) \neq \lambda(2)$.

Therefore condition (ii) is disobeyed.

Derived groups:

$\lambda(B) = \{(1),\ (1,3)(2,6)(5,4)(8,7)\}$;

$\rho(B) = \{(1),\ (1,3)(2,4)(5,7)(6,8)\}$.

Form: $\rho^{-1}(3)\lambda(3)\rho(3) = (3,1)(4,8)(7,2)(6,5) \neq \lambda(3)$.

Therefore condition (iii) is disobeyed.

1.3.3.4. EXAMPLE. Relaxing condition 1.3.1.4. (iv) only.

$$\left|\begin{array}{cc|cc|cc} 1 & 3 & 5 & 7 & 9 & 11 \\ 2 & 4 & 6 & 8 & 10 & 12 \end{array}\right| \xrightarrow{\chi} \left|\begin{array}{cc|cc|cc} 1 & 10 & 5 & 11 & 9 & 7 \\ 12 & 3 & 8 & 2 & 4 & 6 \end{array}\right|$$

χ mixes all panels, therefore condition (i) is obeyed.

$A = \{1,2\} \cong C_2$; not fixed by χ, hence condition (iv) disobeyed.

$B = \{1,3\} \cong C_2$ " " " " " " " "

$S = \{1,5,9\}$; fixed by χ, hence condition (v) obeyed.

Derived groups:

$\lambda(A) = \{(1),\ (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)\}$;

$\rho(A) = \{(1),\ (1,12)(10,3)(5,8)(11,2)(9,4)(7,6)\}$.

Form: $\rho^{-1}(2)\lambda(2)\rho(2) = (12,11)(10,9)(8,7)(6,5)(4,3)(2,1)$

$= \lambda(2)$.

Therefore $\lambda(A)$, $\rho(A)$ centralise each other, hence condition (ii) is obeyed.

Derived groups:

$\lambda(B) = \{(1),\ (1,10)(12,3)(5,11)(8,2)(9,7)(4,6)\}$

$\rho(B) = \{(1),\ (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)\}$.

Form: $\rho^{-1}(3)\lambda(3)\rho(3) = (3,12)(10,1)(7,9)(6,4)(11,5)(2,8)$

$= \lambda(3)$.

Therefore $\lambda(B)$, $\rho(B)$ centralise each other, hence condition (iii) is obeyed.

In this curious example, every left and right coset of A and B is broken up under χ. Hence there exists no χ' obeying condition (iv) yet possessing the same derived groups, $\lambda(A)$, $\lambda(B)$, $\rho(A)$, $\rho(B)$, since each of these have blocks which are cosets of A or B accordingly.

Moreover, in spite of condition (i), G, the orbit of 1 under gp $\{\lambda(A), \lambda(B)\}$ is not the whole of ASB, because the latter is not transitive on ASB. G is in fact the set $\{1,2,7,8,9,10\}$.

1.3.3.5. EXAMPLE. Relaxing condition 1.3.1.4. (v) only.

$$\left|\begin{array}{cc|cc|cc} 1 & 3 & 5 & 7 & 9 & 11 \\ 2 & 4 & 6 & 8 & 10 & 12 \end{array}\right| \xrightarrow{\chi} \left|\begin{array}{cc|cc|cc} 1 & 3 & 9 & 11 & 5 & 7 \\ 2 & 6 & 10 & 12 & 4 & 8 \end{array}\right|$$

χ mixes all panels, therefore condition (i) is obeyed.
A = $\{1,2\} \cong C_2$; fixed by χ, hence condition (iv) obeyed.
B = $\{1,3\} \cong C_2$; " " " (iv) " .
S = $\{1,5,9\}$; not fixed by χ, hence condition (v) <u>not</u> obeyed.
This example has been derived from example 1.3.3.1. simply by interchanging $(AsB)\chi$ and $(As'B)\chi$, where s = 5, s' = 9. This operation leaves the derived groups unaltered. Therefore conditions (ii) and (iii), the A- and B- commuting conditions are still satisfied. However S is no longer fixed by χ. By relaxing condition (v) we permit condition (i) to be satisfied, concealing the fact that G is not the whole of ASB. As in 1.3.3.1., $\{G, \circ\}$ is the dihedral group of order 8.

Examples such as these are straightforward to construct by the methods of chapters 2 and 4. It should be apparent that the conditions (i), (iv) and (v) are easy to apply and to test for. The same cannot be said of conditions (ii) and (iii), the A- and B- commuting conditions. These account for the material content of the problem of constructing all χ , and choosing S, to define a group G = ASB for given groups A, B. The A-commuting condition is the subject of chapter 2, and the simultaneous application of the A- and B-commuting conditions is the subject of chapters 3 and 4.

CHAPTER 2. On the A-commuting Condition.

2.1. For what choice of χ does $\{A, S, B, \chi\}$ satisfy the A-commuting condition?

In the previous chapter we obtained a set of necessary and sufficient conditions for an ordered quadruple, $\{A,S,B,\chi\}$, to define a group (in the sense of 1.2.2.3.).

The most important of these conditions in terms of difficulty of application, namely the A-commuting condition, and by symmetry the B-commuting condition, will be studied in this chapter.

Short of deriving the permutation groups, $\lambda(A)$ and $\rho(A)$, and testing that each element of one commutes with all the elements of the other, it may not appear clear how the A-commuting condition is to be applied, whereas given A, S and B, it is relatively straightforward to establish by inspection whether or not a given permutation, χ, causes the remaining conditions to be satisfied.

We shall furnish a complete description of all $\{A,S,B,\chi\}$ which satisfy the A-commuting condition, in a form which lends itself to the construction of suitable permutations, χ, given A and B. We show in fact that $\{A,S,B,\chi\}$ may be broken down into a number of elementary components, whose structure we shall study in detail. If $\{A,S,B,\chi\}$ defines a group, G, then each of these components corresponds to a double coset, AgA, $g \in G$.

2.1.1. The Transcoset, $\{A, S, \chi\}$, of group A.

We introduce ordered triples, $\{A, S, \chi\}$ to correspond formally with the ordered quadruples, $\{A, S, B, \chi\}$, of chapter 1 for the particular case where B = E, the trivial group. χ is thereby re-interpreted as acting upon the elements of the formal product, AS, rather than AS1; otherwise all the results of chapter 1 still hold true.

Thus, by analogy with definition 1.3.1.1. we may define $\lambda(A)$, $\rho(A)$, the derived groups of $\{A, S, \chi\}$ thus:

2.1.1.1. DEFINITION. For each $a' \in A$, $as \in AS$, the derived groups $\lambda(A)$, $\rho(A)$ of $\{A, S, \chi\}$ are given by:

$$\lambda(a'): as \to (a'a)s,$$
$$\rho(a'): (as)\chi \to ((aa')s)\chi .$$

Accordingly we describe $\{A, S, \chi\}$ as satisfying the A-commuting condition if and only if $\lambda(a)\rho(a') = \rho(a')\lambda(a)$, for all $a, a' \in A$.

2.1.1.2. DEFINITION. A transcoset of group A is an ordered triple, $\{A, S, \chi\}$, where S is a set of indeterminates and χ is a permutation acting upon the elements of the formal product AS, and satisfying the A-commuting condition.

2.1.1.3. DEFINITION. A transcoset of A, $\{A, S, \chi\}$, will be said to be in standard form whenever χ fixes the elements of $1S \subseteq AS$.

NOTE: S does not necessarily contain $1 \in A$, in this chapter.

The actual name "transcoset" was coined to suggest a mapping of a set of right cosets of A, i.e. $\{As \mid s \in S\}$, into left cosets of A, i.e. $\{sA \mid s \in S\}$. The transcoset of A in a group, G, is a mapping $G \rightarrow G$ which describes the embedding of A in G, whereby it furnishes an analogy, albeit imprecise, to the so-called "transfer".

2.1.2. <u>Equivalent transcosets of A.</u>

The important components in the structure of a transcoset, $\{A, S, \chi\}$, when used in the definition of a group, $G \supset A$, are the derived groups, $\lambda(A)$, $\rho(A)$. It is these which describe the multiplication between elements of A and of S in G.

Consequently we shall be interested in pairs of transcosets, $\{A, S, \chi\}$, and $\{A, T, \tau\}$ for which the corresponding derived groups are identical as a result of some 1-1 mapping, γ, of AS onto AT, identifying each element of AS with its image under γ. We shall describe $\{A, S, \chi\}$ as being <u>equivalent</u> to $\{A, T, \tau\}$, for reasons which will become apparent.

If $\{A, S, \chi\}$ is the transcoset of A in a group, G, then S is a right transversal of A in G = AS. Choosing another right transversal, $T \subset G$, will allow another transcoset of A, say $\{A, T, \tau\}$, to be defined for G. Since both describe the same group, G, then equality in G defines a 1-1 mapping $AS \longrightarrow AT$ according to which the derived groups, $\lambda(A)$, $\rho(A)$ are shared by both transcosets. Here then is an example of a pair of equivalent transcosets.

2.1.2.1. DEFINITION. Transcosets $\{A, S, \chi\}$ and $\{A, T, \tau\}$ are <u>identical</u> if and only if every element of S may be identified with an element of T (thereby identifying AS and AT) to yield $\chi = \tau$.

2.1.2.2. DEFINITION. Transcosets $\{A, S, \chi\}$ and $\{A, T, \tau\}$ are <u>equivalent</u> if and only if they share the same derived groups, $\lambda(A)$, $\rho(A)$, under some 1-1 identification of AS and AT.

The process of selecting new right and left transversals of A in AS, (AS)X respectively, to yield a formally-distinct but equivalent transcoset, $\{A,T,\tau\}$ from $\{A,S,\chi\}$ will be called a <u>transformation</u> of $\{A,S,\chi\}$ onto $\{A,T,\tau\}$.

We will show that not only is a transformed transcoset equivalent, by definition, to its original, but that equivalent transcosets may be transformed into identical transcosets. In the process we shall obtain a useful criterion to test for a transcoset (corollary 2.1.2.4.).

2.1.2.3. PROPOSITION. Two given transcosets, $\{A,S,\chi\}$ and $\{A,T,\tau\}$ are equivalent if and only if there exist 1-1 mappings, γ, δ, of AS onto AS with the following properties:

For all $a \in A$, $s_1 \in S$, there exist s, $s' \in S$ and a_s, $\bar{a}_{s'} \in A$ depending on s and s' respectively, such that

γ: $as_1 \longrightarrow a\,a_s s$; (whence γ: $As_1 \longrightarrow As$, and $s_1\gamma = a_s s$)

δ: $(as_1)\chi \longrightarrow (\bar{a}_{s'}\, a\, a')\chi$;

whereupon $\tau = \gamma^{-1}\chi\,\delta$, provided each element of AS is identified with its counterpart in AT via γ, i.e. putting $a{*}t \equiv (a{*}s)\gamma$.

NOTE: The action of γ and δ is to transform $\{A,S,\chi\}$ into an identical transcoset to $\{A,T,\tau\}$, namely the transcoset, $\{A,S\gamma,\gamma^{-1}\chi\delta\}$. We shall show that the above property of γ is necessary and sufficient for the derived group, $\lambda(A)$, to be preserved under the transformation. δ acts on $(AS)\chi$ in an analogous fashion to γ on AS so as to preserve $\rho(A)$.

PROOF. $\{A, S, \chi\}$ and $\{A, T, \tau\}$ are equivalent by 2.1.2.1. if and only if there exists some 1-1 mapping, γ; $AS \rightarrow AT$ such that $\lambda(A)$, $\rho(A)$ are the derived groups of both trans-cosets, where each element in AS is identified with its image under γ in AT.

Furthermore, given any pair of permutations on AS, e.g. $\gamma^{-1}\chi$ and τ, δ can always be found to yield $\tau = \gamma^{-1}\chi\delta$.

Thus the theorem is proved by finding necessary and sufficient conditions on γ and δ for $\lambda(A)$, $\rho(A)$ to be the derived groups of both $\{A, S, \chi\}$ and $\{A, T, \tau\}$.

$\lambda(A)$ shared by both $\{A, S, \chi\}$ and $\{A, T, \tau\}$ is equivalent to: for all $a' \in A$, $\lambda(a')$: $a_1s_1 \rightarrow a'a_1s_1$, for all $a_1 \in A$, $s_1 \in S$; and also

$$\lambda(a'):\ a_2t_2 \rightarrow a'a_2t_2,\ \text{or}$$

$$\lambda(a'):\ (a_2s_2)\gamma \rightarrow (a'a_2s_2)\gamma\ ,\ \text{for}$$

all $a_2 \in A$, $s_2 \in S$.

Equating a_1s_1 with $(a_2s_2)\gamma$; $\lambda(A)$ shared is equivalent to:

$$\gamma \text{ simultaneously maps:} \begin{cases} a_2s_2 \rightarrow a_1s_1, \\ a'a_2s_2 \rightarrow a'a_1s_1, \text{ for all } a' \in A. \end{cases}$$

Putting $a'a_2 = a$ we get: γ: $as_2 \rightarrow a(a_2^{-1}a_1)s_1$, for all $a \in A$.

where $(a_2^{-1}a_1)$ may be chosen to depend upon s_1, and the relationship $s_2 \rightarrow s_1$ is a 1-1 mapping of $S \rightarrow S$.

Hence γ is of the form described if and only if $\lambda(A)$ is shared by $\{A, S, \chi\}$ and $\{A, T, \tau\}$.

Similarly, for all $a' \in A$, $\rho(a')$ shared by both $\{A, S, \chi\}$

and $\{A, T, \tau\}$ is equivalent to:

$\rho(a'): (a_1s_1)\chi \longrightarrow (a_1a's_1)\chi$;

$\rho(a'): (a_2t_2)\tau \longrightarrow (a_2a't_2)\tau$ or

$\rho(a'): (a_2s_2)\gamma(\gamma^{-1}\chi\delta) \longrightarrow (a_2a's_2)\gamma(\gamma^{-1}\chi\delta)$ or

$\rho(a'): (a_2s_2)\chi\delta \longrightarrow (a_2a's_2)\chi\delta$;

for all $a_1, a_2 \in A$, $s_1, s_2 \in S$.

Equating $(a_1s_1)\chi$ with $(a_2s_2)\chi\delta$, $\rho(A)$ shared is equivalent to:

$$\delta \text{ simultaneously maps: } \begin{cases} (a_2s_2)\chi \longrightarrow (a_1s_1)\chi\,, \\ (a_2a's_2)\chi \longrightarrow (a_1a's_1)\chi\,, \text{ for} \end{cases}$$

all $a' \in A$.

Putting $a_2a' = a$ we get: $\delta: (as_2)\chi \longrightarrow ((a_1a_2^{-1})as_1)\chi$,

for all $a \in A$, $s_2 \in S$, where $(a_1a_2^{-1})$ may be chosen to depend on s_1, and the relationship $s_2 \longrightarrow s_1$ is a 1-1 mapping of $S \longrightarrow S$.

Hence δ is of the form described if and only if $\rho(A)$ is shared by $\{A, S, \chi\}$ and $\{A, T, \tau\}$.

2.1.2.4. COROLLARY. An ordered triple, $\{A, S, \chi\}$, is a transcoset if it can be transformed into a known transcoset by a choice of γ, δ, satisfying the conditions of 2.1.2.3.

PROOF. Both $\{A, S, \chi\}$ and its transformed image share $\lambda(A)$ and $\rho(A)$ as their respective derived groups. If the transformed image is a transcoset then $\lambda(A)$ and $\rho(A)$ satisfy the A-commuting condition.

Hence $\{A, S, \chi\}$ is a transcoset also.

The mappings, γ and δ describe a general transformation of $\{A, S, \chi\}$ into $\{A, S\gamma, \gamma^{-1}\chi\delta\}$, which may then be shown to be identical to some $\{A, T, \tau\}$ by identifying T with $S\gamma$. Note that γ has the property:

$$\gamma: as \rightarrow a(s\gamma) \equiv at \in AT, \text{ for all } as \in AS.$$

It represents the selection of a new right transversal of A in AS, identified with T, to make a new formal product, AT, with the same $\lambda(A)$.

δ performs a similar task in $(AS)\chi$, replacing the existing left transversal, $S\chi$ by $S\chi\delta$, to preserve $\rho(A)$.

The process of transforming a transcoset is a major tool in the development of transcoset theory, because a transcoset of general form can sometimes be converted to one of straightforward structure, as the following example will demonstrate.

2.1.2.5. EXAMPLE. Transformation of a transcoset of $A \cong C_5$.

Consider the ordered triple, $\{A, S, \chi\}$, where $A = \{1,2,3,4,5\}$. Let $3 = 2^2$, $4 = 2^3$, $5 = 2^4$.

(AS:)

$$\begin{array}{ccccc} 11 & 12 & 13 & 14 & 15 \\ 21 & 22 & 23 & 24 & 25 \\ 31 & 32 & 33 & 34 & 35 \\ 41 & 42 & 43 & 44 & 45 \\ 51 & 52 & 53 & 54 & 55 \end{array} \xrightarrow{\chi} \begin{array}{ccccc} 11 & 12 & 13 & 14 & 15 \\ 22 & 33 & 24 & 45 & 41 \\ 43 & 44 & 55 & 21 & 52 \\ 54 & 25 & 31 & 32 & 23 \\ 35 & 51 & 42 & 53 & 34 \end{array}$$

$S = S\chi$ is the set $\{11,12,13,14,15\}$. Note that T is also a transversal of A both in AS and $(AS)\chi$, where T is the set $\{51,52,13,14,35\}$.

We select γ to satisfy conditions 2.1.2.3., such that:

$$(11)\gamma = 51,\ (12)\gamma = 52,\ (13)\gamma = 13,\ (14)\gamma = 14,\ (15)\gamma = 35.$$

Likewise we select δ such that $s\gamma = s\delta$, for all $s \in S$.

We thus obtain $\{A, T, \tau\}$ in standard form:

(AT):

$$\begin{array}{ccccc} 51 & 52 & 13 & 14 & 35 \\ 11 & 12 & 23 & 24 & 45 \\ 21 & 22 & 33 & 34 & 55 \\ 31 & 32 & 43 & 44 & 15 \\ 41 & 42 & 53 & 54 & 25 \end{array} \xrightarrow{\tau} \begin{array}{ccccc} 51 & 52 & 13 & 14 & 35 \\ 12 & 23 & 24 & 45 & 11 \\ 33 & 34 & 55 & 21 & 22 \\ 44 & 15 & 31 & 32 & 43 \\ 25 & 41 & 42 & 53 & 54 \end{array}$$

This is identical to the transcoset $\{A, U, \omega\}$:

(AU):

$$\begin{array}{ccccc} 11 & 12 & 13 & 14 & 15 \\ 21 & 22 & 23 & 24 & 25 \\ 31 & 32 & 33 & 34 & 35 \\ 41 & 42 & 43 & 44 & 45 \\ 51 & 52 & 53 & 54 & 55 \end{array} \xrightarrow{\omega} \begin{array}{ccccc} 11 & 12 & 13 & 14 & 15 \\ 22 & 23 & 24 & 25 & 21 \\ 33 & 34 & 35 & 31 & 32 \\ 44 & 45 & 41 & 42 & 43 \\ 55 & 51 & 52 & 53 & 54 \end{array}$$

identifying elements in corresponding positions in the arrays (AT), (AU). $\{A, U, \omega\}$ is a transcoset, as may be shown by the methods of subsection 2.2.1. (in particular proposition 2.2.1.4.). It follows that $\{A, S, \chi\}$ is also a transcoset.

2.1.2.6. PROPOSITION. Every transcoset, $\{A, S, \chi\}$, is equivalent to a transcoset in standard form.

PROOF. By 1.2.1.1. a common transversal, T, of A exists. Hence choose γ, δ such that $S\gamma = S\chi\delta = T$, satisfying 2.1.2.3.

2.1.3. Interpreting $\{A,S,B,\chi\}$ as a transcoset of either A or B.

Let $\{A,S,B,\chi\}$ define a group, G, over ASB, where all symbols take the meaning they had in chapter 1.

Since $\{A,S,B,\chi\}$ satisfies both the A-commuting condition and the B-commuting condition, there are two closely associated transcosets, of A and B respectively. These are $\{A,(SB),\chi\}$ and $\{B,(SA),\chi^{-1}\}$, where the formal product, A(SB) is identified with ASB and the formal product, B(SA), is identified with (ASB)χ . χ and χ^{-1} are re-interpreted inside the ordered quadruples accordingly.

Under this identification, $\{A,(SB),\chi\}$ and $\{A,S,B,\chi\}$ share the same derived groups, $\lambda(A)$ and $\rho(A)$. Hence $\{A,(SB),\chi\}$ is a transcoset, because the A-commuting condition is obeyed. Similarly $\{B,(SA),\chi^{-1}\}$ will be a transcoset related to $\{B,S,A,\chi^{-1}\}$ which by corollary 1.3.2.13. defines the same group, G.

Since $\{A,S,B,\chi\}$ is so closely associated with a transcoset of A and a transcoset of B, we shall describe it as a "generalised" transcoset of A and B.

2.1.3.1. DEFINITION. A generalised transcoset of A and B is an ordered quadruple, $\{A,S,B,\chi\}$, where A and B are finite groups, $A \cap B = E$, S is the union of E and a set of indeterminates (S finite), χ is a permutation defined on ASB, the formal product of A, S and B, satisfying the A- and B-commuting conditions.

Recall that not every generalised transcoset, $\{A,S,B,\chi\}$, defines a group over ASB. It does if and only if:

(i) χ fixes $A11 \cup 1S1 \cup 11B$, (1.3.1.4. (iv) and (v));

(ii) for no $S' \subset S$ does $(AS'B)\chi = AS'B$, (1.3.1.4. (i));

as shown in chapter 1. Of the examples of non-groups exhibited in subsection 1.3.3., 1.3.3.1, 1.3.3.4. and 1.3.3.5. are generalised transcosets.

2.1.3.2. DEFINITION. $\{A,(SB),\chi\}$, $\{B,(SA),\chi^{-1}\}$ will be called the associated transcosets of the generalised transcoset, $\{A,S,B,\chi\}$.

The problem of constructing all generalised transcosets of A and B reduces to finding $|S|$ and χ such that $\{A,(SB),\chi\}$, $\{B,(SA),\chi^{-1}\}$ are transcosets. Furthermore, if the resulting $\{A,S,B,\chi\}$ then satisfies (i) and (ii) above it defines a group.

2.1.4. Partitioning a transcoset into elementary transcosets.

We have shown in 2.1.2. that every transcoset of A is equivalemt to a transcoset in standard form. Thus, without loss of generality, we may confine our discussion of defining groups by transcosets to transcosets in standard form.

Let $\{A,T,\tau\}$ be a transcoset in standard form.
For $g = at \in AT$ let $g\lambda(A) = (at)\lambda(A) = t\lambda(A)$. It is the right coset of A in AT containing g. $g\rho(A)$ will be used

similarly for the left coset of A in $(AT)\chi$ containing g.

2.1.4.1. PROPOSITION. Given any transcoset, $\{A,T,\tau\}$, the subset $t\lambda(A)\rho(A) = t\rho(A)\lambda(A) \subseteq AT$, for any $t \in T$, is mapped onto itself by τ.

PROOF. ~~Let~~ Consider $t\rho(a)\lambda(a_1) \in t\rho(A)\lambda(A)$, for some $t \in T$.
Let $t\rho(a) = t'\lambda(a')$ for some $t' \in T$, $a' \in A$.
Then $t' = t\rho(a)\lambda^{-1}(a')$.

$$\begin{aligned}
\text{Now } (t\rho(a)\lambda(a_1))\tau &= (t\rho(a))\tau\,\rho(a_1) \\
&= (t'\lambda(a'))\tau\,\rho(a_1) \\
&= t'\tau\,\rho(a')\rho(a_1) \\
&= t'\rho(a')\rho(a_1), \text{ since } \tau \text{ fixes } T, \\
&= t\rho(a)\lambda^{-1}(a')\rho(a')\rho(a_1) \\
&= t\rho(a)\rho(a')\rho(a_1)\lambda^{-1}(a'), \text{ (A-commuting con.)} \\
&= t\rho(aa'a_1)\lambda(a'^{-1}) \in t\rho(A)\lambda(A).
\end{aligned}$$

Since τ is 1-1, this means that $(t\rho(A)\lambda(A))\tau = t\rho(A)\lambda(A)$.

2.1.4.2. PROPOSITION. For $t,t' \in T$, $t\rho(A)\lambda(A)$ and $t'\rho(A)\lambda(A)$ are either equal or disjoint.

PROOF. Suppose for $a_1,a_2,a_3,a_4 \in A$,

$$t\rho(a_1)\lambda(a_2) = t'\rho(a_3)\lambda(a_4).$$

Then $t\rho(a_1)\lambda(a_2)\lambda(A)\rho(A) = t'\rho(a_3)\lambda(a_4)\lambda(A)\rho(A)$.
By the A-commuting condition, this reduces to:

$$t\lambda(A)\rho(A) = t'\lambda(A)\rho(A).$$

2.1.4.3. PROPOSITION. $t\rho(A)\lambda(A)$ consists of entire right cosets of A, also of entire left cosets of A.

PROOF. By the A-commuting condition, $t\rho(A)\lambda(A) = t\lambda(A)\rho(A)$.
$t\rho(A)\lambda(A) = \{g\lambda(A) \mid g \in t\rho(A)\}$, a set of entire right cosets of A;
$t\lambda(A)\rho(A) = \{g\rho(A) \mid g \in t\lambda(A)\}$, a set of entire left cosets of A.

2.1.4.4. PROPOSITION. $t\rho(A)\lambda(A)$ is the smallest set consisting of entire right cosets to be mapped onto itself by τ.

PROOF. $t\rho(A)\lambda(A)$ contains the right coset $t\lambda(A) = At$. The smallest set of entire right cosets closed under τ and containing At must therefore contain $(At)\tau$. It is $(At)\tau\lambda(A) = t\rho(A)\lambda(A)$.

Since $t\rho(A)\lambda(A)$ consists of entire right cosets of A in AT then $t\rho(A)\lambda(A) = A\underline{T}$, where $\underline{T} = T \cap t\rho(A)\lambda(A)$. Since also $(A\underline{T})\tau = A\underline{T}$, the ordered triple $\{A,\underline{T},\tau\}$, is defined, where τ is confined to $A\underline{T}$. The derived groups of $\{A,\underline{T},\tau\}$ are $\lambda(A)$, $\rho(A)$ confined to $A\underline{T}$. Thus the A-commuting condition is satisfied and $\{A,\underline{T},\tau\}$ is a transcoset.

2.1.4.5. DEFINITION. An <u>elementary transcoset</u>, $\{A,T,\tau\}$, is one for which $t\rho(A)\lambda(A)$ is the whole of AT, for all $t \in T$.

We have shown so far that every transcoset of A in standard form may be partitioned into one or more distinct elementary transcosets which are disjoint. Conversely, we may concatenate transcosets of A to form larger transcosets as we shall now show.

2.1.4.6. DEFINITION. The <u>concatenation</u> of two given transcosets, $\{A,T_1,\tau_1\}$, $\{A,T_2,\tau_2\}$, where $T_1 \cap T_2 = \Phi$, is the ordered triple, $\{A,T,\tau\}$, where $T = T_1 \cup T_2$, and τ is defined by: For all $a \in A$, $(at)\tau = \begin{cases}(at)\tau_1 \text{ where } t \in T_1, \\ (at)\tau_2 \text{ where } t \in T_2.\end{cases}$

2.1.4.7. PROPOSITION. The concatenation of two given transcosets of A, $\{A,T_1,\tau_1\},\{A,T_2,\tau_2\}$, where $T_1 \cap T_2 = \Phi$, is itself a transcoset.

PROOF. The derived groups, $\lambda_1(A), \rho_1(A)$ of $\{A,T_1,\tau_1\}$, and $\lambda_2(A)$, $\rho_2(A)$ of $\{A,T_2,\tau_2\}$, may be considered as $\lambda(A)$, $\rho(A)$ of $\{A,T,\tau\}$ confined to AT_1, AT_2, respectively, because $AT_1 \cap AT_2 = \Phi$.

If for all $a,a' \in A$, $\lambda_1(a)$ commutes with $\rho_1(a')$,
$\lambda_2(a)$ " " $\rho_2(a')$,
then $\lambda(a)$ " " $\rho(a')$.

Thus the A-commuting condition is satisfied. Hence $\{A,T,\tau\}$ is a transcoset.

2.1.4.8. COROLLARY. An ordered triple, $\{A,T,\tau\}$, is a transcoset of A if and only if it is equivalent to the repeated concatenation of one or more elementary transcosets of A.

We have now reduced the problem of constructing all transcosets of A to that of constructing all elementary transcosets of A. We attend to this problem in section 2.2.

2.2. The Structure of an Elementary Transcoset, $\{A, T, \tau\}$.

In this section we examine the basic structure of an elementary transcoset, with a view to specifying a number of alternative means of constructing all elementary transcosets with certain given properties. We shall also show that every elementary transcoset of A may be embedded in the transcoset of A in some group containing A.

Our main tool will be that of equivalence between transcosets, namely the technique of expressing the transcoset in terms of new left or right transversals of A in AT.

The most important result will be to show that there are surprisingly few non-equivalent elementary transcosets of a given group, A. For example, if A is prime-cyclic, then there is only one non-equivalent elementary transcoset which is not a singlet (i.e., one for which $|T| = 1$).

Quite apart from their application to group theory, elementary transcosets are worthy of study in their own right. It is this conviction which motivates the present section.

Unless otherwise stated, we shall assume throughout this section that every elementary transcoset is in standard form.

2.2.1. The sets of permutations, $\pi(T)$, $\gamma(A)$.

Suppose we are given an elementary transcoset, $\{A,T,\tau\}$, in standard form. T is therefore a common transversal of A.

2.2.1.1. DEFINITION. For all $a \in A$, $t \in T$, define mappings

$$\nu(a)\colon T \to T; \quad \pi(t)\colon A \to A;$$

such that $(at)\tau = a\pi(t)\, t\nu(a) \in AT$.

2.2.1.2. PROPOSITION. For all $a \in A$, $t \in T$, $\nu(a)$ and $\pi(t)$ are permutations.

PROOF. Since T and A are finite sets we simply have to show that $\nu(a)$ and $\pi(t)$ are 1-1.

(i) If $\nu(a)$ is not 1-1 then there exist $t_1, t_2 \in T$ such that

$$t_1\nu(a) = t_2\nu(a), \text{ yet } t_1 \neq t_2.$$

Now: $t_1\rho(a) = (at_1)\tau = a\pi(t_1)\, \underbrace{t_1\nu(a)} = a_1t_3 = t_3\lambda(a_1);$

$t_2\rho(a) = (at_2)\tau = a\pi(t_2)\, \overbrace{t_2\nu(a)} = a_2t_3 = t_3\lambda(a_2);$

for some $a_1, a_2 \in A$, $t_3 \in T$.

Thus: $t_3 = t_1\rho(a)\lambda(a_1^{-1}) = t_1\lambda(a_1^{-1})\rho(a)$

$= t_2\rho(a)\lambda(a_2^{-1}) = t_2\lambda(a_2^{-1})\rho(a)$,

by the A-commuting condition. Cancelling $\rho(a)$ we obtain:

$$t_1\lambda(a_1^{-1}) = t_2\lambda(a_2^{-1}), \text{ or } a_1^{-1}t_1 = a_2^{-1}t_2.$$

This is impossible for $t_1 \neq t_2$, because T is a right transversal of A.

(ii) If $\pi(t)$ is not 1-1 then there exist $a_1, a_2 \in A$ such that

$$a_1\pi(t) = a_2\pi(t), \text{ yet } a_1 \neq a_2.$$

Now: $t\rho(a_1) = (a_1t)\tau = \underbrace{a_1\pi(t)}\, t\nu(a_1) = a\, t_1 = t_1\lambda(a\ );$

$t\rho(a_2) = (a_2t)\tau = \overbrace{a_2\pi(t)}\, t\nu(a_2) = a\, t_2 = t_2\lambda(a\ );$

for some $t_1, t_2 \in T$, $a \in A$.

Thus: $t = t_1\lambda(a)\rho(a_1^{-1}) = t_1\rho(a_1^{-1})\lambda(a)$

$= t_2\lambda(a)\rho(a_2^{-1}) = t_2\rho(a_2^{-1})\lambda(a)$, by the A-commuting conditions. Cancelling $\lambda(a)$ we obtain:

$$t_1\rho(a_1^{-1}) = t_2\rho(a_2^{-1}), \text{ or } (a_1^{-1}t_1)\tau = (a_2^{-1}t_2)\tau,$$

$$\text{or } a_1^{-1}t_1 = a_2^{-1}t_2, \text{ since } \tau \text{ is 1-1.}$$

Since T is a right transversal of A, this is possible if and only if $t_1 = t_2$,

whereupon $a_1 = a_2$, contradicting the assumption.

2.2.1.3. PROPOSITION. $\nu(A)$ is transitive on T.

PROOF. For an elementary transcoset, $AT = t\rho(A)\lambda(A) \ni t'$, for all $t,t' \in T$. Thus there exist $a,a' \in A$ such that

$$\begin{aligned} t' = t\rho(a)\lambda(a') &= (at)\tau\,\lambda(a') \\ &= (a\pi(t)\ t\nu(a))\lambda(a') \\ &= a'(a\pi(t))\ t\nu(a) \\ &= t\nu(a), \text{ since for all } a_1,a_2 \in A, \end{aligned}$$

$t_1,t_2 \in T$, $a_1t_1 = a_2t_2$ implies both $a_1 = a_2$ (= 1 in this case) and $t_1 = t_2$.

Therefore, for all $t,t' \in T$ there exists $a \in A$ such that

$$\nu(a): t \to t'.$$

2.2.1.4. PROPOSITION. If $\{A,T,\tau\}$ is a transcoset then for all $a,a' \in A$, $t \in T$,

(i) $\nu(a'a) = \nu(a')\nu(a)$, (whence $\nu(A)$ is a representation of A)

(ii) $(a'a)\pi(t) = a'\pi(t)\ a\pi(t')$, where $t' = t\nu(a')$.

Conversely, given any ordered triple, $\{A,T,\tau\}$, where A is a group, T is a set of indeterminates, τ is a permutation of AT fixing $1T = T$, if (i) and (ii) hold true for all $a,a' \in A$, $t \in T$, then $\{A,T,\tau\}$ is ~~a~~ an elementary transcoset.

PROOF. The following identity holds for the standard form ordered triple as described, whether or not it is a transcoset: for all $a \in A$, $t \in T$,

(1): $$t\rho(a) = (at)\tau = a\pi(t)\; t\nu(a) = t\nu(a)\lambda(a\pi(t)).$$

Because $\rho(A)$ is a right representation of A we have:

$$t\rho(a'a) = t\rho(a')\rho(a).$$

Substituting for both sides as in identity (1):

(2): $$t\nu(a'a)\lambda((a'a)\pi(t)) = t\nu(a')\lambda(a'\pi(t))\;\rho(a)$$

(3): $$= (t\nu(a'))\rho(a)\;\lambda(a'\pi(t)),$$

by the A-commuting condition,

$$= t\nu(a')\nu(a)\lambda(a\pi(t\nu(a')))\;\lambda(a'\pi(t))$$

substituting as in (1) again,

(4): $$t\nu(a'a)\lambda((a'a)\pi(t)) = t\nu(a')\nu(a)\;\lambda\big[a'\pi(t)\;a\pi(t\nu(a'))\big],$$

since $\lambda(A)$ is anti-isomorphic to A.

Now, in general, for all $a_1, a_2 \in A$, $t_1, t_2 \in T$,

$$t_1\lambda(a_1) = t_2\lambda(a_2) \text{ if and only if } a_1 = a_2 \text{ and } t_1 = t_2.$$

Therefore we may equate elements of A, and elements of T from (4) to give: for all $t \in T$, $a, a' \in A$,

$$\begin{cases} t\nu(a'a) = t\nu(a')\nu(a), \\ (a'a)\pi(t) = a'\pi(t)\;a\pi(t\nu(a')), \end{cases}$$

which yields identities (i) and (ii) of the proposition from the assumption of the A-commuting condition.

Conversely relation (4) above may be obtained from identities (i) and (ii), for all $t \in T$, $a, a' \in A$.

Relation (3) follows by use of (1) from (4).

Relation (2) follows directly from identity (1).

Hence from (2) and (3):

$$t\nu(a')\rho(a)\lambda(a'\pi(t)) = t\nu(a')\lambda(a'\pi(t))\rho(a).$$

Setting $t' = t\nu(a')$ and $a'' = a'\pi(t)$ we obtain:

for all $t' \in T$, $a, a'' \in A$, $t'\rho(a)\lambda(a'') = t'\lambda(a'')\rho(a)$.

Therefore assuming (i) and (ii) implies the A-commuting condition, whence $\{A, T, \tau\}$ is a transcoset.

2.2.1.5. DEFINITION. Let $\underline{\lambda}(A)$ be the left regular representation of A in A. Let $\underline{\rho}(A)$ be the right regular representation of A in A.

2.2.1.6. PROPOSITION. The condition 2.2.1.4.(ii) is equivalent to the condition that, for all $a \in A$, all $t \in T$,

$$\pi(t\nu(a)) = \underline{\lambda}(a)\pi(t)\underline{\lambda}(a^*), \text{ where } a^* = [a\pi(t)]^{-1}.$$

PROOF. For all $a, a' \in A$, all $t \in T$, the following relations are equivalent to each other:

2.2.1.4. (ii): $(aa')\pi(t) = a\pi(t)\ a'\pi(t\nu(a))$;

$$a'\underline{\lambda}(a)\pi(t) = a'\pi(t\nu(a))\underline{\lambda}(a\pi(t));$$
$$a'\underline{\lambda}(a)\pi(t)\underline{\lambda}^{-1}(a\pi(t)) = a'\pi(t\nu(a));$$
$$\underline{\lambda}(a)\pi(t)\underline{\lambda}(a^*) = \pi(t\nu(a)), \text{ where } a^* = [a\pi(t)]^{-1}.$$

2.2.1.7. PROPOSITION. Condition 2.2.1.4. (ii) is obeyed if there exists $t_1 \in T$ such that for all $a \in A$ there exists $a^* \in A$ such that $\pi(t_1\nu(a)) = \underline{\lambda}(a)\pi(t_1)\underline{\lambda}(a^*)$.

PROOF. We need only prove the proposition for all $t \in T$ in place of t_1, because if a^* exists it is equal to $(a\pi(t))^{-1}$. This follows from the fact that for all $a \in A$, $\pi(t)$ and $\pi(t\nu(a))$ both fix the element 1. Thus:

$$1 = 1\underline{\lambda}(a)\pi(t)\underline{\lambda}(a^*) = a\pi(t)\underline{\lambda}(a^*) = a^*(a\pi(t)).$$

For all $t \in T$ there exists $a' \in A$ such that $t = t_1\nu(a')$. Hence, from the assumed property of t_1, for all $t \in T$, $a \in A$, there exists a^{**} such that:

$$\begin{aligned}\pi(t\nu(a)) &= \pi(t_1\nu(a')\nu(a)) = \pi(t_1\nu(a'a))\\ &= \underline{\lambda}(a'a)\pi(t_1)\underline{\lambda}(a^{**})\\ &= \underline{\lambda}(a)\underline{\lambda}(a')\pi(t_1)\underline{\lambda}(a^{**}).\end{aligned}$$

By assumption, $\pi(t) = \pi(t_1\nu(a')) = \underline{\lambda}(a')\pi(t_1)\underline{\lambda}(a'*)$,

for some $a'* \in A$. Hence:

$$\pi(t\nu(a)) = \underline{\lambda}(a)\pi(t)\,\underline{\lambda}^{-1}(a'*)\underline{\lambda}(a**).$$

Therefore $a*$ exists to prove the proposition, given by

$$\underline{\lambda}(a*) = \underline{\lambda}^{-1}(a'*)\underline{\lambda}(a**),$$

or, as shown above, by $a* = [a\pi(t)]^{-1}$, because $\pi(t)$ fixes 1 for all $t \in T$.

Conditions (i) and (ii) of proposition 2.2.1.4. may thus be restated as follows:

2.2.1.4. There exists $t_1 \in T$ such that for all $a, a' \in A$,

(i): $\nu(a'a) = \nu(a')\nu(a)$,

(ii): $\pi(t_1\nu(a)) = \underline{\lambda}(a)\pi(t_1)\underline{\lambda}(a*)$, for some $a* \in A$.

2.2.1.8. THEOREM. Given a group, A, a permutation, σ, on set A fixing 1, and a transitive representation, $\nu(A)$ of A, acting on a set, T, of indeterminates, then defining $\pi(T)$ by:

$$\begin{cases} \pi(t_1) = \sigma \text{ for some } t_1 \in T, \\ \pi(t) = \underline{\lambda}(a)\sigma\underline{\lambda}^{-1}(a\sigma) \text{ for each } t \in T, \text{ where } a \in A \text{ is such that} \end{cases}$$

$\nu(a)$ maps t_1 onto t; yields a unique elementary transcoset, $\{A, T, \tau\}$, where π and ν define τ by 2.2.1.1. provided that $\pi(t_1\nu(a)) = \pi(t_1)$ if $t_1\nu(a) = t_1$, for all $a \in A$.

PROOF. We need to show that τ, defined by

$$(at)\tau = a\pi(t)\ t\nu(a), \text{ for all } a \in A,\ t \in T,$$

is in fact a 1-1 mapping of $AT \rightarrow AT$.

Now for all $t \in T$ there exists $a \in A$ such that $t = t_1\nu(a)$, for some given t_1.

Therefore $\pi(t)$ is uniquely defined as a permutation on A fixing 1, for all $t \in T$, by:

$$\pi(t_1) = \sigma,$$

$$\pi(t_1\nu(a)) = \underline{\lambda}(a)\sigma\underline{\lambda}^{-1}(a\sigma), \text{ for all } a \in A,$$

provided that $t_1\nu(a) = t_1$ implies $\pi(t_1\nu(a)) = \pi(t_1)$.

Hence τ is defined for all $a \in A$, $t \in T$, as:

$$(at)\tau = a\pi(t)\ t\nu(a);$$

and π, ν satisfy 2.2.1.4. (i) and (ii), in the form restated above.

Therefore by 2.2.1.4., $\{A,T,\tau\}$ exists and is a transcoset.

2.2.2. <u>Collapsed Elementary Transcosets.</u>

Given an elementary transcoset in standard form, $\{A,T,\tau\}$, we note that the largest number of elements which T can contain is $|A|$. This corresponds to the largest degree of $\nu(A)$ possible for it to be transitive on T. If $|T| = |A|$ then $\nu(A)$ must be permutation-isomorphic to the right regular representation of A in A.

2.2.2.1. DEFINITION. A <u>collapsed</u> elementary transcoset, $\{A,T,\tau\}$, is one for which $|T| < |A|$.

Collapsing ~~implies~~ means that for some $t \in T$ and $1 \neq a \in A$,

$$t = t\nu(a).$$

This furthermore obviously implies that $\pi(t) = \pi(t\nu(a))$, or that: $\pi(t) = \underline{\lambda}(a)\pi(t)\underline{\lambda}^{-1}(a\pi(t))$, from 2.2.1.6.

Therefore only those elementary transcosets can be collapsed, or collapsed further, for which there exist $t, t' \in T$, $t \neq t'$ yet $\pi(t) = \pi(t')$. If this condition does not pertain then the elementary transcoset will be called <u>fully collapsed</u> or <u>uncollapsible</u>.

2.2.2.2. DEFINITION. A collapsible elementary transcoset is one for which there exist $t, t' \in T$, $t \neq t'$ yet $\pi(t) = \pi(t')$.

2.2.2.3. DEFINITION. An elementary transcoset, $\{A, \underline{T}, \underline{\tau}\}$, is the collapsed image of an elementary transcoset, $\{A, T, \tau\}$, if there exists a mapping, μ, possibly many-one, where $\mu: T \rightarrow \underline{T}$ relates the respective $\pi(T)$, $\nu(A)$; $\underline{\pi}(\underline{T})$, $\underline{\nu}(A)$ thus: for all $a \in A$, $t \in T$, $(t\mu)\underline{\nu}(a) = (t\nu(a))\mu \in \underline{T}$; and for some given $t_1 \in T$, $\pi(t_1) = \underline{\pi}(t_1\mu)$.

2.2.2.4. PROPOSITION. To every collapsible elementary transcoset there exists a unique fully collapsed elementary transcoset which is its collapsed image.

PROOF. Let $\{A, T, \tau\}$ be a collapsible transcoset, with representation $\nu(A)$ as defined in 2.2.1.1., and $\pi(t_1) = \sigma$, for some $t_1 \in T$.

Let μ be defined by $t\mu = t'\mu$ if and only if $\pi(t) = \pi(t')$, for all $t, t' \in T$.

Let $\underline{\nu}(A)$ be defined by $(t\mu)\underline{\nu}(a) = (t\nu(a))\mu$, for all $a \in A$, $t \in T$.

Now $\underline{\nu}(A)$ is 1-1 on $\underline{T}$, which is equivalent to stating that

$(t\mu)\underline{\nu}(a) = (t'\mu)\underline{\nu}(a)$ if and only if $t\mu = t'\mu$.

Now LHS: $(t\nu(a))\mu = (t'\nu(a))\mu$,

therefore $\pi(t\nu(a)) = \pi(t'\nu(a))$,

$$\underline{\lambda}(a)\pi(t)\underline{\lambda}(a^*) = \underline{\lambda}(a)\pi(t')\underline{\lambda}(a'^*),$$

$$\pi(t) = \pi(t')\underline{\lambda}(a^{**}), \text{ where } a^{**} = a^{*-1}a'^{*}.$$

Since both $\pi(t)$ and $\pi(t')$ fix 1, $\underline{\lambda}(a^{**}) = (1)$, whence $\pi(t) = \pi(t')$. Therefore $t\mu = t'\mu$.

Hence $\underline{\nu}(A)$ has been defined as a representation of $\nu(A)$, and hence of A, on the elements of $\underline{T} = T\mu$.
By theorem 2.2.1.8, σ, $\underline{T}$ and $\underline{\nu}(A)$ define a unique elementary transcoset. This is fully collapsed, and the collapsed image of $\{A,T,\tau\}$.

2.2.3. <u>A Transcoset of group A in Sym(A).</u>

There are a number of useful results which we must derive for an elementary transcoset in standard form, $\{A,T,\tau\}$, in particular concerning the set of permutations, $\pi(T)$, associated with it. We have shown, by theorem 2.2.1.8., that for any given permutation, σ, on the elements of A fixing 1, there exists an elementary transcoset, $\{A,T,\tau\}$, in standard form having $\pi(t) = \sigma$ for some $t \in T$.

We shall show in this subsection that any elementary transcoset can be embedded in a group transcoset, because then we shall be able to use group theory to describe the internal structure of an elementary transcoset, yet without making explicit reference to any particular group in which it may be embedded.

As the first step towards this goal we shall embed in a group transcoset any given uncollapsible elementary transcoset of A, that is, for which the permutations of $\pi(T)$ are all distinct. The group concerned will be Sym(A), the symmetric group on the elements of set A.

First we must exhibit a generalised transcoset defining

Sym(A). Let $G = AB$, $A \cap B = E$ be a Zappa product of groups A and B. Let $\{A,B,\chi\}$ be a transcoset of A in G in standard form. This exists because B is a common transversal of A in G. This is the associated transcoset of the generalised transcoset, $\{A,E,B,\chi\}$ defining G, which it yields immediately.

2.2.3.1. DEFINITION. Let $\pi(B)$ be the union of all $\pi(T)$, $T \subseteq B$ being a right transversal of an elementary transcoset embedded in $\{A,B,\chi\}$.

2.2.3.2. PROPOSITION. $\pi(B)$ is anti-isomorphic to B.

PROOF. Every left coset of B in G can be expressed as aB, $a \in A$.

For all $b \in B$, $ba = (ab)\chi = a\pi(b)\, b\gamma(a)$.

Therefore $b(aB) = a\pi(b)\, B$, i.e. $\pi(b)\colon aB \rightarrow baB$.

$\pi(B)$ is the representation of B on the left cosets of B in G.

A well-known theorem (see for example W.R. Scott (1964) p 374) states that any permutation group, Σ, with a transitive subgroup, Λ, factorises into $\Lambda\Sigma_x = \Sigma_x\Lambda$, where Σ_x is the stabiliser of any point, x. Hence Sym(A) factorises into $\lambda(A)$ and $\mathrm{Sym}_1(A)$, the stabiliser of point 1 in set A. $\lambda(A)$ and $\mathrm{Sym}_1(A)$ are fully inconjugate in Sym(A), since $\lambda(A)$ is regular on A but $\mathrm{Sym}_1(A)$ fixes an element, viz. 1. This proves:

2.2.3.3. PROPOSITION. Given any group, A, there exists $B \cong \mathrm{Sym}_1(A)$ such that $\mathrm{Sym}(A) \cong G = AB = BA$, $A \cap B = E$.

For such a group, $G = AB$, the set of permutations, $\pi(B)$ is a representation of $B \cong Sym_1(A)$ on the elements of A fixing 1. $\pi(B)$ is faithful if and only if no subgroup of B is normal in G. Now a subgroup of $Sym_1(A)$ cannot be normal in Sym(A) since the only normal subgroup of Sym(A) is $Alt(A) \not\subset Sym_1(A)$, $|A| \geq 5$. Hence $\pi(B)$ is faithful. It must itself be equal to $Sym_1(A)$.

As shown in section 2.1.4., $\{A,B,\chi\}$ can be partitioned into elementary transcosets of the form $\{A,T,\tau\}$, $T \subseteq B$. Since $\pi(B)$ is faithful, for no non-equal $b,b' \in B$ does $\pi(b) = \pi(b')$. Therefore all the elementary transcosets are fully collapsed. Moreover, by theorem 2.2.1.8., every fully collapsed elementary transcoset appears precisely once in the partition of $\{A,B,\chi\}$, since $\pi(B)$ contains every permutation, σ, of the elements of set A fixing 1 precisely once. We may state this result as a theorem:

2.2.3.4. THEOREM. Every fully collapsed elementary transcoset in standard form of A can be embedded in the transcoset of a group; in particular, of A in Sym(A).

The requirement to be in standard form is because otherwise collapsibility is not defined.

We shall have need of a more general result, namely that every elementary transcoset in standard form can be embedded in the transcoset of a group. Since every transcoset is equivalent to one in standard form, this will imply furthermore that every elementary transcoset can be embedded in the transcoset of a group.

Note that the group concerned cannot be chosen as the symmetric group, since the only elementary transcosets of A embeddable in Sym(A) are those which are equivalent to fully collapsed elementary transcosets. In fact:

2.2.3.5. PROPOSITION. There exist collapsible elementary transcosets which are equivalent to no fully collapsed transcoset.

For a proof of this we indicate example 2.2.6.3. below.

2.2.3.6. THEOREM. The collapsible elementary transcoset in standard form, $\{A,T,\tau\}$, can be embedded in a group transcoset, in particular in the transcoset of A in ~~the~~ some Zappa product $F = AC$.

NOTE. Embedding $\{A,T,\tau\}$ in some $\{A,C,\chi\}$, a transcoset in standard form of a group $F = AC$, $C \supset T$, ensures that $\pi(C)$ cannot be a faithful representation of C because there exist non-equal $t,t' \in T \subset C$ such that $\pi(t) = \pi(t')$. Thus some subgroup, N, of C must be normal in F.

PROOF. We shall construct F as an extension of some group, N by Sym(A) such that $N \lhd F$.
Let $\nu(A)$, $\pi(T)$ be defined for the given $\{A,T,\tau\}$. $\nu(A)$ is permutation isomorphic to a representation on the right cosets of some subgroup, H, of A, because it is a transitive representation of A.
Let $G = AB \cong \mathrm{Sym}(A)$, where $B \cong \mathrm{Sym}_1(A)$, as introduced

for theorem 2.2.3.4.

Let N be any finite group containing a complex, Q, $|Q| = |T|$, such that G is represented by ~~an~~ a subgroup of Aut(N) automorphism~~-group~~, $\nu_N(G)$ defined on N with the following property:

Q is a set of transitivity of $\nu_N(A) \subseteq \nu_N(G)$, restricted to which $\nu_N(A)$ is permutation isomorphic to $\nu(A)$.

Such an N may be chosen as the direct product of (G:H) copies of C_2, the group of order 2. Then any permutation of the set R, $|R| = (G:H)$, of non-trivial elements lying inside each of the (G:H) direct factors of N defines an automorphism of N. Now define the permutation group, $\nu_N(G)$, on R to be permutation isomorphic to the representation of G on the right cosets of H in G. Then there exists $Q \subseteq R$ corresponding to set $\{Ha \mid a \in A\}$, on which $\nu_N(A)$ is permutation isomorphic to $\nu(A)$.

Let $\mu: Q \to T$ be the 1-1 mapping describing the permutation isomorphism: $\nu_N(A) \to \nu(A)$.

Define F as a semi-direct product of N by G. F exists, and is uniquely specified by the composition:

$(gn)(g'n') = (gg')(n^{g'}n')$, for all $g, g' \in G$, $n, n' \in N$, where $n \to n^g$ defines some automorphism group of N induced by G, which we choose to be $\nu_N(G)$. Thus: $n^g = g^{-1}ng = n\nu_N(g)$.

Since N is normal in F it follows that $C = BN = NB$ is a subgroup of F.

Since also $G \cap N = E$, then for no $a \in A$, $b \in B$, $n \in N$ not all equal to 1 does $ab = n$, i.e. $a = nb^{-1}$. Hence $A \cap C = E$.

Thus F is the Zappa product of A and C as well as of G and N.

C is the Zappa product (also semi-direct product) of B and N.

Let $\{A,C,\chi\}$ be the transcoset in standard form of A in $F = AC$. It may be partitioned into elementary transcosets. $\{A,B,\chi\}$ exists as a group transcoset (of G) embedded in $\{A,C,\chi\}$. As $\{A,B,\chi\}$ has embedded in it every fully collapsed elementary transcoset of A, it contains a collapsed image of the given transcoset, $\{A,T,\tau\}$ Such a (fully) collapsed image always exists, by 2.2.2.4. Let it be $\{A,S,\chi\}$. Thus for some $t \in T$ there exists $b \in S \subseteq B$ such that $\pi(b) = \pi(t)$.

Let $n \in Q \subseteq N$ be such that $t = n\mu$, where μ is defined above. Consider that elementary transcoset, $\{A,U,\chi\}$ in the partition of $\{A,C,\chi\}$, such that $bn \in U \subset C$.

Let its associated representation of A be $\nu_U(A)$.

We now show that $\{A,T,\tau\}$ and $\{A,U,\chi\}$ are identical, by identifying T and U as follows:

$$(n\mu =)\ t \rightarrow bn;$$

$$t\nu(a) \rightarrow (bn)\nu_U(a), \text{ for all } a \in A.$$

Since N is normal in F which is homomorphic to G,

$$\pi(bn) = \pi(b) = \pi(t).$$

Hence by 2.2.1.8., we simply have to show that $\nu_U(A)$ is permutation-isomorphic to $\nu(A)$ under the above point-mapping, $(\rightarrow)$.

Now, for each $b'n' \in U$, $(a(b'n'))\chi = (b'n')a$

$$= a\pi(b'n')\ (b'n')\nu_U(a), \text{ for all } a \in A.$$

But $(b'n')a = b'a(a^{-1}n'a) = b'a\ n'\nu_N(a)$

$$= a\pi(b')\ b'\nu_S(a)\ n'\nu_N(a),$$

where $\{A,S,\chi\}$ is the fully collapsed image of $\{A,T,\tau\}$

where $S \subset B$, and $\nu_S(A)$ is the associated representation of A on S.

Hence for all $b'n' \in U$, all $a \in A$,

$(b'n')\nu_U(a) = (b'\nu_S(a))(n'\nu_N(a))$.

It follows that ν_U is permutation isomorphic to ν_N under $b'n' \rightarrow n'$, and thence to ν under μ:

$b'n' \rightarrow n' \rightarrow (n'\mu = t')$, for all $b'n' \in U$, provided that $n = n\nu_N(a)$ implies $b = b\nu_S(a)$, for all $a \in A$.

We show this as follows:

$n = n\nu_N(a)$ implies $t = t\nu(a)$, where $t = n\mu$, by definition of μ;

which implies $b = b\nu_S(a)$, since $\{A,S,\chi\}$ is the collapsed image of $\{A,T,\tau\}$.

Thus we may identify T with $U \subset C$ in such a way that $\{A,T,\tau\}$ is embedded in $\{A,C,\chi\}$, i.e. in a group transcoset.

The signåficance of theorems 2.2.3.4. and 2.2.3.6. is that we are now able to use group theory to describe the structure of any elementary transcoset.

When the elementary transcoset, $\{A,T,\tau\}$, is embedded in a transcoset of the group $G \supseteq A$, the derived groups, $\lambda(A)$, $\rho(A)$, become the left and right regular representations of A in G, restricted to AT. Thus we have for all $a,a' \in A$, $t \in T$,

$(at)\lambda(a') = a'at$,

$(at)\rho(a') = ata'$,

$(at)\tau = (t\tau)\rho(a) = ta$ (if in standard form),

$(at)\tau\rho(a') = taa'$ (")

juxtaposition of elements, including the formal product, at, representing products in the group, G, which need not itself be specified.

Using this device of products in unspecified groups containing A, our investigation of the structure of an elementary transcoset can employ more concise notation and shorter, more immediate proofs than proceeding by first principles, i.e. by making use of the derived groups and the A-commuting condition to determine constraints on τ.

2.2.4. The five types of elementary transcoset.

In subsection 2.2.3. we showed how every elementary transcoset of A may be embedded in a group transcoset, $\{A,C,\chi\}$, of A in F = AC. We now investigate the detailed structure of any standard form elementary transcoset, $\{A,T,\tau\}$, by supposing it to be embedded in some group transcoset, $\{A,C,\chi\}$ where $C \supseteq T$, but the transcoset is otherwise unspecified. As a result of doing so, for all $a,a' \in A$, $t \in T$, the term $t\lambda(a)\rho(a') = t\rho(a')\lambda(a)$ may be written as ata', where juxtaposition represents multiplication in the unspecified group AC.

The structure we shall uncover is, however, a property purely of $\{A,T,\tau\}$, and is independent of whatever group the transcoset, $\{A,T,\tau\}$, may be embedded in.

2.2.4.1. PROPOSITION. Let $u = \bar{a}t$ be any element of AT. The set, H_u, of all $a \in A$ such that $ua = a'u$, for some $a' \in A$, is a subgroup of A. The corresponding set $\{a'\}$ also forms a subgroup, J_u, of A isomorphic to H_u.

NOTE. This implies that $uH_u = J_u u$, for all $u \in AT$.

PROOF. If $a_1, a_2 \in H_u$ then there exist $a_1', a_2' \in A$ such that

$$ua_1 = a_1'u; \quad ua_2 = a_2'u.$$

Now $u(a_1a_2) = a_1'ua_2 = (a_1'a_2')u$.

Hence by definition, $(a_1a_2) \in H_u$. H_u is closed under multiplication, hence it is a subgroup of A.

Now we see that u induces a product-preserving 1-1 mapping of H_u into A, i.e. onto the image J_u. Hence J_u is a subgroup of A isomorphic to H_u.

2.2.4.2. COROLLARY. For all $t \in T$, $H_t = \{a \in A \mid t\gamma(a) = t\}$,

$$J_t = H_t\pi(t).$$

PROOF. By definition of π and γ, for all $a \in A$, $t \in T$,

$$ta = (at)\tau = a\pi(t)\ t\gamma(a),$$

$$= a\pi(t)\ t \text{ if and only if } a \in H_t.$$

Therefore $a \in H_t$ if and only if $t\gamma(a) = t$.

Furthermore $ta = a\pi(t)\ t$ is equivalent to: $a\pi(t) \in J_t$.

Therefore $a\pi(t) \in J_t$ if and only if $a \in H_t$.

2.2.4.3. DEFINITION. For all $u \in AT$, H_u, J_u are called <u>H- and J-subgroups</u>, respectively, of A in $\{A,T,\tau\}$.

As a result of 2.2.4.2. we may say that $\gamma(A)$ is permutation-isomorphic to the representation of A on the right cosets of H_t in A, for any $t \in T$.

We now ask how H_u is related to $H_{u'}$, and J_u is related to $J_{u'}$, for any pair of elements, $u,u' \in AT$. We shall show that in each case they are conjugate.

2.2.4.4. PROPOSITION. For all $u,u' \in AT$, H_u is related to $H_{u'}$ and J_u is related to $J_{u'}$ in the following way:
Let $a,a' \in A$ be such that $u' = aua'$.(They exist because $AT = AuA$ for an elementary transcoset).
Then $H_{u'} = \{h' = a'^{-1}ha' \mid h \in H_u\} = a'^{-1}H_ua'$;

$$J_{u'} = \{j' = aja^{-1} \mid j \in J_u\} = aJ_ua^{-1};$$

and $u'h' = j'u'$ whenever $uh = ju$.

PROOF. For all $h \in H_u$, $u'(a'^{-1}ha') = auha'$, because $u' = aua'$,

$= ajua'$, where $ju = uh$, $j \in J_u$,

$= (aja^{-1})u'$, because $u' = aua'$.

Hence by definition, $a'^{-1}ha' \in H_{u'}$, and $aja^{-1} \in J_{u'}$,

where $uh = ju$.

Hence $H_{u'} \supseteq a'^{-1}H_ua'$, $J_{u'} \supseteq aJ_ua^{-1}$.
Equality may be shown by pursuing the argument with u,u' interchanged. Hence all H- and J-subgroups are of the same order, and

$$H_{u'} = a'^{-1}H_ua', \quad J_{u'} = aJ_ua^{-1}.$$

We now wish to distinguish five types of elementary transcoset, based on whether or not an H-subgroup and corresponding J-subgroup are conjugate in A. Such a classification into types is important in the investigation of equivalent elementary transcosets. We shall show that all equivalent elementary transcosets are of the same type, so that expressing the elementary transcoset in terms of a different common transversal does not change the type.

2.2.4.5. DEFINITION. Given an elementary transcoset, $\{A,T,\tau\}$, we obtain H_u and $J_u \subseteq A$ for some $u \in AT$. If one of the following sets of conditions hold the elementary transcoset is said to be of the corresponding type:

- $H = J \triangleleft A$ if and only if $H_u = J_u \neq E$ is a proper normal subgroup of A;
- $H \sim J$ if and only if H_u and J_u are proper non-normal subgroups of A which are equal or conjugate in A;
- $H \not\sim J$ if and only if H_u and J_u are subgroups of A which are neither equal nor conjugate in A;
- $H = E$ if and only if $H_u = E = \{1\}$, whereupon $J_u = H_u = E$;
- $H = A$ if and only if H_u is the whole of A, whereupon $J_u = H_u = A$.

2.2.4.6. PROPOSITION. The type of an elementary transcoset is independent of whatever element $u \in AT$ is chosen to test for the above conditions.

PROOF. Let $H \sim J$ signify that H is conjugate to J in A. Take each type in turn.

$H = J \lhd A$: Since $H_u \sim H_{u'}$, $J_u \sim J_{u'}$ for all $u,u' \in AT$, then the fact that $H_u = J_u \lhd A$ for a given $u \in AT$ means that for all $u' \in AT$, $H_{u'} = H_u = J_u = J_{u'} \lhd A$.

$H \sim J$: Since for all $u,u' \in AT$, $H_u \sim H_{u'}$, $J_u \sim J_{u'}$, then $H_u \sim J_u$ implies that any pair of subgroups from the set of all H- and J-subgroups are conjugate in A.

$H \not\sim J$: Since conjugacy is transitive, the assertion that for some $u \in AT$, H_u and J_u are not conjugate implies that no H-subgroup can be conjugate to a J-subgroup of A in A.

$H = E$: All H- and J-subgroups of A must be trivial. Such an elementary transcoset has $|T| = |A|$.

$H = A$: All H- and J-subgroups of A must consist of the whole of A. Then $|T| = 1$.

2.2.4.7. COROLLARY. The five types of elementary transcoset are mutually exclusive and exhaustive, i.e. an elementary transcoset can be of one and only one type.

2.2.4.8. PROPOSITION. Two equivalent elementary transcosets must be of the same type.

PROOF. If the elementary transcosets, $\{A,T,\tau\}$ and $\{A,U,\omega\}$ are equivalent, then from subsection 2.1.2., U may be identified with a right transversal of A in AT so that AT and AU both refer to the same double coset in some unspecified host group, $AuA = A(a't)A = AtA$, where

$u = a't$ for some $u \in U$, $a' \in A$, $t \in T$.

Thus $H_u = H_{(a't)}$, and $J_u = J_{(a't)}$.

Therefore $\{A,T,\tau\}$ and $\{A,U,\omega\}$ are of the same type.

In the next subsection we explore the extent to which the converse of proposition 2.2.4.8. is true.

2.2.5. <u>The conditions for two elementary transcosets of the same type to be equivalent.</u>

In subsection 2.2.4. we defined the type of an elementary transcoset in terms of the corresponding H- and J-subgroups of A, and showed that equivalent elementary transcosets were of the same type.

In this subsection we will determine when two elementary transcosets, $\{A,T,\tau\}$, $\{A,U,\omega\}$, of the same type are equivalent. In particular we shall show that any two type: $H = E$ elementary transcosets of A are equivalent, and that if H_t possesses no (outer) automorphisms not induced by elements of A, then the corresponding elementary transcoset may be transformed to a standard form in which $\pi(t)$ is the identity permutation, for all $t \in T$, provided it is not of type $H \neq J$. Such an elementary transcoset is of extremely simple construction, since its fully collapsed image is the "trivial" transcoset, $\{A,S_o,\chi_o\}$, where $|S_o| = 1$ and $\chi_o = (1)$.

2.2.5.1. LEMMA. Given an elementary transcoset, $\{A,T,\tau\}$, in standard form, $\pi(t)$ maps entire right cosets of H_t in A onto entire right cosets of J_t, subject to $(ha)\pi(t) = h\pi(t)a\pi(t)$, $h \in H_t$, $a \in A$. Furthermore, $\pi(t)$ defines an isomorphism of $H_t \rightarrow J_t$.

PROOF. From 2.2.1.4., for all $h \in H_t$, $a \in A$ (whence $h\pi(t) \in J_t$),

(1): $(ha)\pi(t) = h\pi(t)\, a\pi(t\nu(h)) = h\pi(t)\, a\pi(t)$.

Hence $(H_t a)\pi(t) = J_t(a\pi(t))$, i.e. $\pi(t)$ maps a right coset,

$H_t a$, onto a right coset, $J_t(a\pi(t))$, for all $a \in A$.

Furthermore, putting $a \in H_t$ in equation (1) shows $\pi(t)$ to define an isomorphism: $H_t \rightarrow J_t$, because $\pi(t)$ fixes 1.

2.2.5.2. PROPOSITION. Two elementary transcosets of the same type, $\{A,T,\tau\}$ and $\{A,U,\omega\}$, are equivalent if for some $t \in T$, $u \in U$,

$H_t = H_u$; $J_t = J_u$; and for all $h \in H_t$, $j \in J_t$,

$uh = ju$ whenever $th = jt$.

PROOF. From lemma 2.2.5.1. we see that $\pi(t)$ maps entire right cosets of H_t onto entire right cosets of J_t in A.
Let $C \subseteq A$ be a common transversal of the right cosets of H_t and the right cosets of J_t in A; i.e. $A = H_tC = J_tC$.
Suppose $C \ni 1$, as the common representative of H_t, J_t.
Now we construct the set $S \subset AT$ using t and C thus:

$$S = \{c^{-1}tc \mid \text{all } c \in C\}.$$

We must first prove that S is a common transversal of A in AT, $(AT)\tau$.
Note that $|S| = |C| = (A:H_t) = |T|$, hence S contains the right number of elements.
Furthermore, no two elements of S lie in the same right coset of A in AT. Otherwise:

for some $s_1, s_2 \in S$, $s_1 = as_2$, $a \in A$,

whence $s_1 = c_1^{-1}tc_1 = ac_2^{-1}tc_2$ for some $c_1, c_2 \in C$.

Therefore $t = (c_1ac_2^{-1})t(c_2c_1^{-1})$,

which from 2.2.4.1. implies $c_2c_1^{-1} \in H_t$, i.e. $c_2 \in H_tc_1$.

This is impossible since C is a right transversal of H_t.

Likewise, no two elements of S lie in the same left coset of A in $(AT)\tau$. Otherwise:

for some $s_1, s_2 \in S$, $s_1 = s_2a$, $a \in A$,

whence $s_1 = c_1^{-1}tc_1 = c_2^{-1}tc_2a$ for some $c_1, c_2 \in C$.

Therefore $t = (c_1c_2^{-1})t(c_2ac_1^{-1})$,

which from 2.2.4.1. implies $c_1c_2^{-1} \in J_t$, i.e. $c_1 \in J_tc_2$.

This is impossible since C is a right transversal of J_t.

Thus we have proven that S is a common transversal of A in AT, $(AT)\tau$.

We now transform $\{A,T,\tau\}$ by replacing T by the common transversal, S, to yield a standard form elementary transcoset, say, $\{A,S,\chi\}$. We know that both describe the same double coset, AtA, in some host group, F. Also, since $S \ni t$, H_t is an H-subgroup of A in $\{A,S,\chi\}$ as well as $\{A,T,\tau\}$.

Let $\pi'(S)$ be defined as in 2.2.1.4. for $\{A,S,\chi\}$.

Taking $s = t$, for all $a = hc \in A = H_tC$,

$$(as)\chi = sa = ta = thc$$

$$= jtc, \text{ for some } j \in J_t,$$

$$= jc(c^{-1}tc) = (jc)s' \text{ for some } s' \in S.$$

(1): Hence $\pi'(s)$: $hc \rightarrow jc$, where $s = t$, $h \in H_t$, $j = h\pi(t) \in J_t$.

In a similar manner, $\{A,U,\omega\}$ may be transformed into $\{A,\underline{S},\underline{\chi}\}$ in which, defining the corresponding $\underline{\pi}(\underline{s})$, $\underline{s} \in \underline{S}$, (2): $\underline{\pi}(\underline{s})$: $hc \rightarrow jc$,

where $\underline{s} = u \in U$ corresponding to $t \in T$ by the assumption:

$H_u = H_t$; $J_u = J_t$; and for all $h \in H_t$, $j \in J_t$,

$uh = ju$ whenever $th = jt$.

It follows that we may choose the same h, j, c, as in (1).

Thus: $\underline{\pi}(\underline{s} = u) = \pi(s = t) = \sigma$, say.

Hence $\{A,S,\chi\}$ and $\{A,\underline{S},\underline{\chi}\}$ are identical, because the corresponding $\nu(A)$ and $\underline{\nu}(A)$ (as defined by 2.2.4.1.) are permutation isomorphic under some 1-1 identification of S and $\underline{S}$ which identifies t and u, since both are permutation isomorphic to the representation of A on the right cosets of $H_t = H_u$ in A, as follows from 2.2.4.2.

Hence, by theorem 2.2.1.8., σ, S, $\nu(A)$ and σ, $\underline{S}$, $\underline{\nu}(A)$ define identical transcosets. Therefore $\{A,T,\tau\}$ and $\{A,U,\omega\}$ are equivalent because they can be transformed into a pair of identical elementary transcosets.

2.2.5.3. PROPOSITION. Two elementary transcosets, $\{A,S,\chi\}$, $\{A,T,\tau\}$, are equivalent if and only if there exists $u \in AS$, $v \in AT$, such that $H_u = H_v$, $J_u = J_v$, and for each $h \in H_v$, $j \in J_v$,

$uh = ju$ whenever $vh = jv$.

PROOF. Equivalent elementary transcosets are equivalent to the same elementary transcoset of A in some host group $F \supseteq A$. Hence there exist $u \in AS$, $v \in AT$, corresponding to the same element in F. Then $H_u = H_v$, $J_u = J_v$, and for each $h \in H_v$, $j \in J_v$, $uh = ju$ whenever $vh = jv$.

Conversely, given $u \in AS$, $v \in AT$ with these properties, one may choose new common transversals, $S' \ni u$, $T' \ni v$, of A in AS, AT respectively, thereby defining a transformation of the respective elementary transcosets into a pair which is shown to be equivalent by proposition 2.2.5.2.

We now consider the implications of proposition 2.2.5.3. for different types of elementary transcoset.

2.2.5.4. PROPOSITION. Any two type: $H = E$ elementary transcosets of A are equivalent.

PROOF. The conditions of proposition 2.2.5.3. are trivially satisfied if all H- and J-subgroups of A are trivial.

In particular, all type: $H = E$ elementary transcosets of A are equivalent to an elementary transcoset of particularly simple structure, namely $\{A,S,\chi_0\}$, where $\nu(A)$ is permutation isomorphic to the right regular representation of A, and $\pi(s) = (1)$ for all $s \in S$.

This provides the means of constructing all type: $H = E$ elementary transcosets, namely from $\{A,S,\chi_0\}$ by transformation.

2.2.5.5. PROPOSITION. Any type: $H \sim J$ elementary transcoset, $\{A,T,\tau\}$ may be transformed into some $\{A,S,\chi\}$, for which

$$(H_t =)\ H_s = J_s \text{ for some } s \in S.$$

PROOF. Type: $H \sim J$ implies that for $t \in T$ there exists $a \in A$ such that $J_t = a^{-1} H_t a$.

By proposition 2.2.4.4. a new common transversal, S containing the element, at, will define a transformation to an equivalent elementary transcoset in standard form, say $\{A,S,\chi\}$, such that for $s = at$,

$$H_s = H_t,$$
$$J_s = aJ_ta^{-1} = H_t \text{ also.}$$

2.2.5.6. PROPOSITION. An elementary transcoset, $\{A,T,\tau\}$, is equivalent to some $\{A,S,\chi_0\}$, for which $\pi(s) = (1)$, all $s \in S$, if and only if for some $t \in T$ there exists $a \in A$ such that the isomorphism $\pi(t)$: $H_t \to J_t$ is also induced by a. I.e. for all $h \in H_t$, $h\pi(t) = a^{-1}ha$.

NOTE. It follows that $\{A,T,\tau\}$ cannot be of type: $H \nleftarrow J$.

PROOF. Such an $\{A,S,\chi_0\}$ exists, defined as in 2.2.1.8. by $\sigma = (1)$, S and $\nu_0(A)$: $S \to S$ permutation isomorphic to the representation of A on the right cosets of H_t in A. Then for some $s \in S$, $H_s = H_t$.

$\pi(s) = (1)$ is true for all $s \in S$ if true for any $s \in S$, by 2.2.1.4. (ii). So for all $h \in H_s$, $h\pi(s) = h$.

Hence $J_s = H_s$ and $sh = hs$, for all $h \in H_s$.

If for $\{A,T,\tau\}$ there exists $a \in A$ such that $a^{-1}ha = h\pi(t)$

$= j \in J_t$, for all $h \in H_t$, then $(at)h = ajt = h(at)$.

Thus, for choice of $u = at$, $H_u = H_t$, $J_u = H_t$,

and for all $h \in H_t$, $uh = hu$.

Hence $\{A,S,\chi_0\}$ and $\{A,T,\tau\}$ are equivalent, as seen by putting $v = s$, $j = h$, to satisfy the conditions of proposition 2.2.5.3.

Conversely if $\{A,T,\tau\}$ and $\{A,S,\chi_0\}$ are equivalent then by 2.2.5.3. there exists $u = at \in AT$ such that $H_u = J_u$, and for all $h \in H_u$, $uh = hu$.

Hence for all $h \in H_u$, $th = a^{-1}uh = a^{-1}hu = (a^{-1}ha)t = jt$. Thus $H_t = H_u$, $J_t = a^{-1}H_u a$, and furthermore the isomorphism induced by t, i.e. $\pi(t)$: $H_t \rightarrow J_t$ is also induced by a.

We have shown in this subsection that the question of whether or not two elementary transcosets are equivalent reduces to whether or not a pair of H- and J-subgroups of A can be found in common which are fused similarly, i.e. for which the isomorphism: $H_t \rightarrow J_t$, induced by an element, t, of the relevant double coset of A in some host group, is the same in each case.

These results considerably reduce the diversity of elementary transcosets of A which need to be considered when attempting to construct all generalised transcosets defining groups containing A.

2.2.6. Fusing any given pair of isomorphic subgroups of A.

The problem now presents itself of whether an elementary transcoset of a given group A may always be found so that any given pair of isomorphic subgroups of A become an H- and corresponding J-subgroup.

B.H. Neumann (1954) has studied the problem of embedding group A in a group G such that any given isomorphism between a pair of subgroups, say C and D in A, is induced by some element $g \in G$. Thus: $C = g^{-1}Dg$. C and D are thereby said to be fused in G. He showed that not only may G always be found, but that if A is finite, G may be chosen as Sym(A).

The latter result is attributed to P. Hall.

The proposition that G may always be found such that for $g \in G$ the elementary transcoset corresponding to AgA has C as an H-subgroup (H_g) and D as a J-subgroup (J_g) is stronger than P. Hall's theorem, which merely asserts that $G \ni g$ exists such that H_g contains C and J_g contains D.

Moreover, the proposition is not true for G = Sym(A). If $A = V_4$, the Klein 4-group, Sym(A) does not contain an element g such that AgA corresponds to an elementary transcoset with $H_g \cong J_g \cong C_2$. Yet, as we shall show, there exists just such an elementary transcoset, with $H_g = C$, $J_g = D$, for any choice of $C \cong D \cong C_2$ in V_4. We shall construct one as an example (2.2.6.3. below). Such an elementary transcoset cannot be embedded in Sym($A = V_4$) in the manner of theorem 2.2.3.4., because all the fully collapsed elementary transcosets of V_4 are of type: H = A , i.e. consist of just one left (hence right) coset of V_4. To establish this, note that any permutation on the non-identity elements of V_4 is an automorphism. Hence

$$V_4 \triangleleft \mathrm{Sym}(V_4).$$

2.2.6.1. THEOREM. To every finite group A having isomorphic subgroups, C, D, there exists an elementary transcoset, $\{A,T,\tau\}$ ~~for which for~~ and $t \in T$, such that $C = H_t$, $D = J_t$, and t induces any given isomorphism: $C \rightarrow D$.

PROOF. By lemma 2.2.5.1., $\pi(t)$ corresponding to $\{A,T,\tau\}$ must map entire right cosets of $H_t = C$ onto entire right cosets of $J_t = D$, inducing some isomorphism: $H_t \to J_t$ or $C \to D$ subject to: $(ca)\pi(t) = c\pi(t)\, a\pi(t)$, $c \in C$, $a \in A$.

Hence choose permutation $\sigma : A \to A$, fixing 1, to satisfy these conditions for any given isomorphism of C onto D.

σ always exists and may be found as follows:

Find a common transversal, R, of right cosets of C and right cosets of D in A. Thus $A = CR = DR$.

Let σ fix R.

For all $c \in C$ let $c\sigma = d \in D$ so as to induce the given isomorphism.

Then $(cr)\sigma = c\sigma\, r\sigma = c\sigma\, r = dr$, for all $c \in C$, $r \in R$, where $d \in D$. This defines a suitable σ.

Now define $\nu(A): T \to T$, where T is a set of indeterminates, to be permutation-isomorphic to the representation of A on the right cosets of C in A. Let the corresponding point-mapping carry the right coset, C itself, onto $t \in T$. Identify σ with $\pi(t)$.

By theorem 2.2.1.8., σ, T and $\nu(A)$ define a unique elementary transcoset, $\{A, T, \tau\}$, with the required properties.

2.2.6.2. COROLLARY. Every finite group, A, having isomorphic subgroups, C, D, may be embedded in a group F such that for some $t \in F$, t induces any given isomorphism of C onto D (i.e. by $C \to tCt^{-1} = D$) and moreover, $D = A \cap tAt^{-1}$.
In other words, C and D are the largest fused subgroups of A under t.

PROOF. The elementary transcoset, $\{A, T, \tau\}$, shown to exist by theorem 2.2.6.1. may be embedded in a group, F, by theorem 2.2.3.4., if fully collapsed, or by theorem 2.2.3.6. if collapsible. Then $t \in T$ has the above property because

$$C = H_t, \; D = J_t.$$

2.2.6.3. EXAMPLE. An elementary transcoset which is equivalent to no fully collapsed elementary transcoset.

Let A be the set $\{1,2,3,4\}$. Define $A \cong V_4$ by means of $\underline{\lambda}(A)$:

$\underline{\lambda}(1) = (1)$; $\underline{\lambda}(2) = (1,2)(3,4)$; $\underline{\lambda}(3) = (1,3)(2,4)$;

$\underline{\lambda}(4) = (1,4)(2,3)$;

Suppose $C = \{1,2\}$; $D = \{1,3\}$. Then $C \cong D \cong C_2$.

Let σ be any permutation on the elements of A fixing 1, satisfying the conditions of lemma 2.2.5.1. For example, $\sigma = (2,3)$ maps entire right cosets of C onto entire right cosets of D.

Choose $T = \{t,u\}$; a set of indeterminates; $|T| = (A:C) = 2$.

Equate σ with $\pi(t)$.

Note that $\underline{\lambda}(a)\sigma\, \underline{\lambda}^{-1}(a\sigma) = \sigma$, for all $a \in A$. Hence $\pi(t) = \pi(u)$.

Define $\nu(A)$: $T \to T$ as being permutation isomorphic to the representation of A on the right cosets of C in A. Thus:

$$\nu(1) = \nu(2) = (1); \quad \nu(3) = \nu(4) = (t,u).$$

Thus, by 2.2.1.8., σ, T, $\nu(A)$ define a unique elementary transcoset, which may be obtained by direct application of definition 2.2.1.1., namely:

$$(AT): \begin{matrix} 1t & 1u \\ 2t & 2u \\ 3t & 3u \\ 4t & 4u \end{matrix} \quad \xrightarrow{\tau} \quad (AT)\tau: \begin{matrix} 1t & 1u \\ 3t & 3u \\ 2u & 2t \\ 4u & 4t \end{matrix}$$

Now all fully collapsed elementary transcosets of V_4 appear in the partition of the transcoset of A in Sym(A), where $A = V_4$. Since A is normal in $Sym(A = V_4)$, they are all of type: $H = A$, whereas the above example is of type: $H \neq J$. Hence the above example, $\{A, T, \tau\}$, is equivalent to no uncollapsible elementary transcoset.

CHAPTER 3. On Zappa Products.

In this chapter we shall consider those standard-form generalised transcosets, $\{A,S,B,\chi\}$ defining groups for the special case where $S = E$.

The group thus defined is of the form $G = AB = BA$, where every $g \in G$ has a unique expression as $g = ab$, or alternatively as $g = b'a'$, for some $a,a' \in A$, $b,b' \in B$. This implies that $A \cap B = E$, otherwise if for a, $a' \in A$, $b,b' \in B$, $a \neq a'$, $b \neq b'$, $ab = a'b'$, then $b'b^{-1} = a'^{-1}a$, exemplifying a non-trivial element common to both A and B.

Such groups are products of two trivially intersecting subgroups and are called Zappa products, as noted in Chapter 1. They have been studied using mainly four distinct techniques:

(i): Generalised extension theories resembling that of Schreier.
(ii): Permutation representations of A (say), on the right cosets of A in $G = AB$.
(iii): Use of Burnside's result on the solvability of G if A and B are both p-groups,
(iv) S-ring theory.

Use of technique (i) has been briefly discussed in the introduction, 1.1. To this ought to be added mention of the work of G. Casadio (1941) for the case where A and B contain isomorphic normal subgroups, and of C. Tibiletti (1957) who studied the embedding of permutable products in the wreath product, as introduced by M. Krasner and L. Kaloujnine (1950, 1951).

Considerable attention has been paid in the literature of permutable products to criteria of non-simplicity, directed towards uncovering stronger solvability criteria for G = AB, given that A and B possess certain combinations of the properties of being: nilpotent; solvable; dihedral; abelian; a p-group. Here attention is drawn to the work of B. Huppert and N. Itô (1953, 1954, 1956). Insofar as they can be said to rely upon any one technique, this would be technique (iii) in the above list, namely to reduce the question of the solvability of G = AB for certain choices of A and B properties to that of the (known) solvability of the product of two prime-power order groups, by induction on the order of G as one of the main tools. Their work shows that if one of the factors of G = AB is nilpotent, and the other has one of several properties, e.g. of being abelian, or of prime-power order, or Hamiltonian, then G is solvable. A further important generalisation has been achieved by Kegel (1961), who proved the solvability of the product of two nilpotent groups.

There can however be no counterpart in permutable product theory to the property of Schreier extensions that the extension, G, of A by B is solvable if and only if both A and B are solvable. The simple groups, Alt(5) and S_{168} are both

Zappa products of solvable groups. Furthermore, in each case one of the solvable factors is nilpotent.

The existence of these two elementary counterexamples to obvious conjectures suggests that the laws governing the existence of non-solvable groups which are Zappa products of solvable factors are not straightforward, nor easily uncovered. A purely group theoretic proof of the solvability of the Zappa product of two odd co-prime order p-groups (Burnside's fundamental result) has only recently been achieved by D.A. Goldschmidt (1970), in spite of considerable attention to the problem by various workers in the past.

The apparent suitability, however, of the study of elementary transcosets of one factor of a Zappa product for providing insight into questions of this nature leads us to propose this as a fifth technique. The aims of employing this technique in addition to the four already listed might be considered as settling the following questions:

- (1) Does the non-existence of simple Zappa products of two p-groups have a counterpart in the non-existence of certain types of elementary transcoset, viz. those which satisfy certain conditions for embeddability in such Zappa products?
- (2) Does the theorem of Feit and Thompson (1963) on the solvability of odd order groups likewise have a counterpart in the prohibition of certain types of elementary transcoset?

In spite of the acknowledged depth of the latter result, question (2) is especially tempting, because odd order Zappa products have (generalised) transcosets, and corresponding elementary transcosets of distinctive structure. If $\{A,B,\chi\}$ is a transcoset of the Zappa product, $G = AB$, then A and B may be partitioned into disjoint subsets, $\underline{A} \cup \underline{A}^{-1} \cup E$, $\underline{B} \cup \underline{B}^{-1} \cup E$ respectively, where $\underline{A}$ and $\underline{B}$ are inverse-free, such that the subsets: $\underline{A}\underline{B}$, $\underline{A}(\underline{B})^{-1}$, $(\underline{A})^{-1}B$, $(\underline{A})^{-1}(\underline{B})^{-1}$ of AB are all mapped onto themselves by χ. An affirmative answer to question (2) would imply an affirmative answer to question (1) for the case of odd order p-groups, leaving only the case of the Zappa product of an odd p-group with a 2-group to be considered,

This technique of studying Zappa products through the elementary transcosets of just one of the factors (e.g. A) has similarities in this respect with S-ring theory; technique (iv) on our list. Attention is drawn to expository accounts in H. Wielandt (1964) and W.R. Scott (1964), and to the work of O. Tamaschke (1966) in deriving a generalised character theory within S-ring theory.

S-rings over a finite group, A, correspond to transcosets of A as follows. Let $G = AB$ be the Zappa product of A and B. There exists an S-ring over A called the <u>double coset S-ring</u> which has as a basis a set of elements in the group algebra, R_A, (or group-ring; see M. Hall, jnr. (1959) p 247 et seq.) which correspond to the distinct complexes, $(BaB) \cap A$, for $a \in A$. These complexes of A also happen to be the transitivity sets of $\pi(B)$: $A \rightarrow A$, as defined for the transcoset of A in AB. However, S-ring theory will not concern us further here.

The two questions posed above will motivate the ensuing treatment of transcosets defining Zappa products. Although sufficiently strong non-existence criteria for given types of transcoset have not yet been found to furnish even partial alternative proofs of either Burnside's result for prime-power factors, or the result of Feit and Thompson, the constructive methods of transcoset theory permit the search for transcosets satisfying the known criteria for embeddability in simple Zappa products of odd order, or of two p-groups. These should then be available for further examination, hopefully leading to stronger criteria. This indicates the use of an electronic computer in a search for counter-examples to tentative conjectures: a pedestrian, but instructive method of conducting research.

One such investigation has been undertaken on the elementary transcosets of D_8, the dihedral group of order 8 which are embeddable in simple Zappa products. Some of the results are discussed in subsection 3.2.3.

The choice of D_8 was prompted by the following considerations:

- D_8 is dihedral, a p-group and therefore also nilpotent. Zappa products of factors with properties selected from these are popular candidates in the literature for non-simplicity criteria.
- D_8 is nonetheless a factor of a simple Zappa product (S_{168}).
- D_8 is moreover a Sylow subgroup of S_{168}. In theories of non-normal extension especial interest attaches to assem-

bling finite groups from their Sylow subgroups.

The fully collapsed elementary transcosets of D_8 were thus suspected to be a rich source of counter-examples to plausible conjectures concerning the elementary transcosets of Zappa products. In particular they offered the means for a closer study of S_{168}, a simple group which appears to have some interesting unique properties, e.g. as shown in B. Huppert (1956).

3.1. Structure of the generalised transcoset of a Zappa product.

In this section we state a number of propositions relating to the structure of an elementary transcoset, $\{A,T,\tau\}$, which is embedded in the transcoset, $\{A,B,\chi\}$, defining the Zappa product, $G = AB$.

These propositions will be of concern in section 3.2., where we exhibit a table of distinct elementary transcosets of D_8, the dihedral group of order 8, which allows us to investigate Zappa products with D_8 as one factor, where no subgroup ($\neq E$) of the other factor is normal in the whole group.

3.1.1. Cycles of χ and the orders of elements in $G = AB$.

Recall that for all $ab \in$ group $G = AB$, $A \cap B = E$, defined by the standard form transcoset, $\{A,B,\chi\}$, $(ab)\chi = ba$. The following is immediate:

3.1.1.1. PROPOSITION. Elements lying in the same cycle of χ are conjugate in G.

PROOF. For all $a \in A$, $b \in B$, $(ab)\chi = ba = b(ab)b^{-1} = a^{-1}(ab)a$.

It follows that all such elements are of the same order. This order bears the following relationship to the length of the χ-cycle containing the elements:

3.1.1.2. PROPOSITION. The length, r, of a cycle of χ divides the order of an element, $ab \in AB$, which it moves.

PROOF. Let $a_1, a_2, \ldots, a_r, a_{r+1}, \ldots,$
$b_1, b_2, \ldots, b_r, b_{r+1}, \ldots,$
be two sequences of (not necessarily non-recurring) elements of A and B respectively, such that, for all $i = 1,2,\ldots,$

$$(a_i b_i)\chi = b_i a_i = a_{i+1} b_{i+1}.$$

If r is the length of the cycle, $(a_1b_1, a_2b_2, \ldots, a_rb_r)$, then $a_{r+1}b_{r+1} = a_1b_1$.

Let n be the order of the element, a_1b_1, in G. Then:

$$\begin{aligned}(a_1b_1)^n = 1 &= a_1(b_1a_1)^{n-1}b_1, \\ &= a_1(a_2b_2)^{n-1}b_1, \\ &= a_1a_2(b_2a_2)^{n-2}b_2b_1, \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \\ &= (a_1a_2 \ldots a_n)(b_nb_{n-1} \ldots b_2b_1).\end{aligned}$$

Since AB is a Zappa product, this implies that:

$$(a_1a_2 \ldots a_n) = 1 = (b_nb_{n-1} \ldots b_1).$$

Now, since $(a_ib_i)\chi = b_ia_i = (a_ib_i)^{a_i} = a_{i+1}b_{i+1}$,

$$(a_1b_1)\chi^n = (a_1b_1)^{a_1a_2 \ldots a_n} = a_1b_1.$$

Since, however, r is the smallest positive integer such that $(a_1b_1)\chi^r = a_1b_1$, then r must divide n.

3.1.1.3. COROLLARY. If $\{A,B,\chi\}$ defines the group, $G = AB$, then $\chi^{|G|} = (1)$.

PROOF. By Lagrange's theorem, the order of an element of G divides the order of G. Hence if r is the length of any given cycle of χ, then r divides $|G|$.

Thus χ is of order dividing $|G|$.

3.1.2. χ and the inverse mapping of G.

Let $\phi_G = \phi_{AB}$ be the inverse mapping of the group, G, defined by the transcoset, $\{A,B,\chi\}$.

Then ϕ_G: $ab \to b^{-1}a^{-1}$, for all $a \in A$, $b \in B$.

Now χ: $ab \to ba$ " " , by definition.

3.1.2.1. DEFINITION. Let θ_{AB}: $ab \to a^{-1}b^{-1}$, for all $a \in A$, $b \in B$.

θ_{AB}, restricted to $A \subset AB$ equals ϕ_A;

θ_{AB}, restricted to $B \subset AB$ equals ϕ_B.

3.1.2.2. PROPOSITION. $\chi^{-1} = \phi_{AB}\theta_{AB}$; $\chi = \theta_{AB}\phi_{AB}$.

PROOF. For all $a \in A$, $b \in B$,

$$(ab)\theta_{AB}\chi = (a^{-1}b^{-1})\chi = b^{-1}a^{-1} = (ab)\phi_{AB}.$$

Now both θ_{AB} and ϕ_{AB} are of order 2. Hence:

$$\chi = \theta_{AB}^{-1}\phi_{AB} = \theta_{AB}\phi_{AB}.$$

Therefore: $\chi^{-1} = \phi_{AB}^{-1}\theta_{AB}^{-1} = \phi_{AB}\theta_{AB}$.

3.1.2.3. PROPOSITION. $\chi^{-1} = \phi_{AB}\chi\phi_{AB} = \theta_{AB}\chi\theta_{AB}$.

PROOF. $\phi_{AB}\chi\phi_{AB} = \phi_{AB}(\theta_{AB}\phi_{AB})\phi_{AB} = \phi_{AB}\theta_{AB} = \chi^{-1}$.

$\theta_{AB}\chi\theta_{AB} = \theta_{AB}(\theta_{AB}\phi_{AB})\theta_{AB} = \phi_{AB}\theta_{AB} = \chi^{-1}$.

3.1.2.4. PROPOSITION. χ consists of:

(i) cycles of length 1,

(ii) pairs of cycles of length greater than 1, permuting pairs of inverse complexes of G,

(iii) cycles of even length each permuting a self-inverse complex of G.

If G is odd order, then χ consists of (i) and (ii) only.

PROOF. If χ contains a cycle: $(g_1, g_2, \ldots, g_r)$, then

$= \phi_G^{-1}\chi^{-1}\phi_G$ contains the cycle:

$$\phi_G^{-1}(g_r, g_{r-1}, \ldots, g_1)\phi_G$$
$$= (g_r\phi_G, g_{r-1}\phi_G, \ldots, g_1\phi_G)$$
$$= (g_r^{-1}, g_{r-1}^{-1}, \ldots, g_1^{-1}).$$

These two cycles of χ must be either identical or disjoint.
If disjoint then $g_1, g_2, \ldots, g_r$; $g_r^{-1}, g_{r-1}^{-1}, \ldots, g_1^{-1}$ are two inverse complexes. The pair of cycles permuting these are of type (ii).

If identical then $g_1, g_2, \ldots, g_r$ is a self-inverse complex of G. Call it C. If any $g_1 \in C$ is its own inverse then it is of order 2, whereupon by 3.1.1.2. $|C| = 1$ or 2. If 1 then the corresponding cycle is of type (i). Otherwise pair off the elements of C with their own inverses, so each is paired with a distinct element, showing $|C|$ to be even. The corresponding cycle is then of type (iii). By 3.1.1.2. each element lying in C must be of even order.

It follows that if G is of odd order then there can be no type (iii) cycles.

We now consider the inverse structure of G in terms of elementary transcosets of A.

If AgA, $g \in G \supseteq A$, is a double coset then the set of all inverses of elements lying in AgA is equal to the double coset $Ag^{-1}A$. For every element $aga' \in AgA$, the inverse is $a'^{-1}g^{-1}a^{-1} \in Ag^{-1}A$.

3.1.2.5. DEFINITION. A self-inverse double coset of A in a group G is one which contains the inverse of any (hence all) of its elements.

3.1.2.6. PROPOSITION. If A is of odd order then a self-inverse double coset of A must contain an element of order 2.

PROOF. A double coset consists of $(A:H_t)$ right cosets of A for some subgroup, H_t, of A. Hence if A is of odd order then AgA will contain an odd number of elements.
If furthermore AgA contains no element of order 2 then pairing every element of self-inverse AgA with its inverse will be impossible since this requires there to be an even number of elements.

3.1.2.7. PROPOSITION. If a type: $H \not\sim J$ elementary transcoset of A, $\{A, T, \tau\}$, is embedded in the transcoset of A in some host group, G, then the double coset, AT, cannot be self-inverse.

PROOF. If AT is self-inverse then for $t \in T$, $t^{-1} = at'$ for some $a \in A$, $t' \in T$.
From the definition of H- and J- subgroups, $H_t = J_{t^{-1}}$; $J_t = H_{t^{-1}}$. By proposition 2.2.4.4. all the H-subgroups are conjugate. Hence $H_t \sim H_{t^{-1}} = J_t$. Thus the elementary transcoset cannot be of type $H \not\sim J$.

3.1.2.8. COROLLARY. If a type: $H \not\sim J$ elementary transcoset of A is embedded in a group transcoset then there must exist a separate $H \not\sim J$ elementary transcoset of A corresponding to the inverse double coset of A.

The pair of elementary transcosets, $\{A,T,\tau\}$, $\{A,U,\omega\}$, corresponding to inverse double cosets in some group $G \supsetneq A$ will be of the same type and have the same cycle-structure by proposition 3.1.2.4., but are not necessarily equivalent. Thus if for $g \in AT$, $H_g \lhd A$ but $J_g \ntriangleleft A$, then for $g^{-1} = f \in AU$, $H_f \ntriangleleft A$, $J_f \lhd A$. By 2.2.4.4., 2.2.4.6., this is true for all $f \in AU$, which violates a necessary condition for equivalence (2.2.5.3.), viz. that for some $f \in AU$, $g \in AT$, $H_f = H_g$, etc.

3.1.3. Constructing the Generalised Transcoset of a Zappa product.

In order to construct all generalised transcosets, $\{A,S,B,\chi\}$, defining a group, $G = ASB$, from elementary transcosets of A and B, the suggestion has been put forward to build up the associated transcosets, $\{A,(SB),\chi\}$ and $\{B,(SA),\chi^{-1}\}$ by in-step concatenation of selected elementary transcosets of A and of B respectively.

In the case of a Zappa product (i.e. where $S = E$) a much simpler process can be used. Under the restriction that no non-trivial subgroup of B will be normal in the resulting group $G = AB$, it suffices to assemble $\{A,(SB),\chi\} = \{A,B,\chi\}$ alone by concatenation of elementary transcosets in standard form selected so that $\pi(B)$, the union of $\pi(T)$ contributed by each elementary transcoset, $\{A,T,\chi\}$, is a faithful (anti-isomorphic) representation of B.

We shall say that $\{A,B,\chi\}$ <u>defines</u> the Zappa product, $G = AB$, It does this through yielding directly $\{A,E,B,\chi\}$ as a generalised transcoset of A and B, which defines G in the strict sense of 1.2.2.3. We shall prove this as a theorem below.

Zappa products may be constructed in the most general case by the same method initially, to form $G' = AB'$, and then by Schreier extension theory to form $G = AB$, where $G \triangleright N \triangleleft B$ such that $B' = B/N$.

3.1.3.1. THEOREM. Let $\{A,B,\chi\}$ be a transcoset of A in standard form. B is given as the union of E and a set of indeterminates. Let $\pi(B)$ be the union of all $\pi(T)$ for each $T \subset B$, where $\{A,T,\chi\}$ is an embedded elementary transcoset.
If $\pi(B)$ forms a permutation group, with $\pi(b) \neq \pi(b')$ for each pair of distinct $b,b' \in B$ then $\{A,B,\chi\}$ defines a Zappa product, $G = AB$, where $\pi(B)$ constitutes a faithful anti-isomorphic representation of subgroup B.

PROOF. Since by assumption of properties of $\pi(B)$, $\{A,B,\chi\}$ is the concatenation of distinct fully collapsed elementary transcosets of A, it can be embedded in $A,\underline{B}$, ; defining group $A\underline{B} = \mathrm{Sym}(A)$ as in subsection 2.2.3.; such that $B \subseteq \underline{B} = \mathrm{Sym}_1(A)$.
Now $\pi(\underline{B})$ is an anti-isomorphic representation of $\underline{B}$, of which $\pi(B)$ is a subgroup. Hence B, represented by $\pi(B)$, is a subgroup of $\underline{B}$ such that $AB = BA$. Hence AB is a subgroup of $A\underline{B}$, and is uniquely defined by $\{A,B,\chi\}$ through $\{A,E,B,\chi\}$, the generalised transcoset defining $G = AB$.

We note also the following proposition, which provides a useful simplicity criterion for a transcoset defining a Zappa product.

3.1.3.2. PROPOSITION. If a Zappa product, $G = AB$, of co-prime order subgroups, A and B, has a normal subgroup, $K \neq E$, then there exist subgroups, $L \lhd A$, $M \lhd B$, at least one of which is proper, such that $K = LM = ML$; $LB = BL$; $MA = AM$.

PROOF. Let η be the homomorphism of $G = AB$ onto $G' = A'B'$ with kernel K. Since $|G'| < |G|$, where $|G'| = |A'| \times |B'|$; $|G| = |A| \times |B|$; $|A|$, $|B|$ co-prime, it follows that $|A'| < |A|$ or $|B'| < |B|$. Then the mappings η: $A \rightarrow A'$; η: $B \longrightarrow B'$; must be homomorphisms with kernels, $L \lhd A$, $M \lhd B$, say, at least one of which is a proper subgroup of A or B respectively.
Therefore $L = K \cap A$; $M = K \cap B$.
Now $LM \subseteq K$ is a set of distinct elements, since every expression of $g \in G$ as $g = ab$, $a \in A$, $b \in B$, is unique. Thus $LM = K$ because:

$$|LM| = |L| \times |M| = (|A|/|A'|) \times (|B|/|B'|) = |G|/|G'| = |K|.$$

Similarly $ML = K$.
Now for all $b \in B$, $bK = Kb = bML = LMb$, because $K \lhd G$.
Thus $BML = LMB$, or $BL = LB$, because $BM = MB = B$.
Similarly $AK = KA = ALM = MLA = AM = MA$.

3.1.3.3. COROLLARY. The Zappa product defined by transcoset $\{A, B, \chi\}$, A and B of co-prime order, is simple if and only if for no proper normal subgroup, $L \lhd A$, or $M \lhd B$, does χ map $LB \subset AB$, or $AM \subset AB$, onto themselves.

3.2. The Generation of Fully Collapsed Elementary Transcosets.

3.2.1. Method of investigating the transcoset of A in Sym(A) by computer.

An ALGOL program was written to print out the fully collapsed standard form elementary transcosets of a given low order finite group A by examining Sym(A) was written and run on a Burroughs B6700 computer for the case $A = D_8$, the dihedral group of order 8.

The elementary transcosets forming the output, say $\{A,T,\tau\}$, are unique to automorphism of A and to permutation of the elements of T. Since, as we have shown, every fully collapsed elementary transcoset of A in standard form appears precisely once in the transcoset of A in G = AB, where $B = Sym_1(A)$, the program embodies the following algorithm:

3.2.1.1. ALGORITHM. To obtain the distinct fully collapsed elementary transcosets of A in standard form, unique to automorphism of A and mutually non-identical.

(1) Enumerate the permutations of $Sym_1(A)$. Suppose the next permutation in sequence is σ.

(2) Generate the set of permutations $\pi(t\nu(A)) = \lambda(a)\sigma\lambda^{-1}(a\sigma)$ for all $a \in A$, forming $T = t\nu(A)$ at the same time by identifying $t\nu(a)$ and $t\nu(a')$ whenever $\pi(t\nu(a)) = \pi(t\nu(a'))$ $\{A,T,\tau\}$, thus defined, will be fully collapsed.

(3) If for any $t' \in T$, $\pi(t')$ occurs earlier in the enumeration of $Sym_1(A)$ then $\{A,T,\tau\}$ has been examined already. Discard σ and go to (1).

(4) If any $\alpha^{-1}\pi(t')\alpha$, $t' \in T$, $\alpha \in Aut(A)$ (the group of auto-

morphisms of A), occurs earlier in the enumeration of $Sym_1(A)$, then $\{A,T,\tau\}$ is of the same abstract structure as an elementary transcoset examined already, i.e. where A has been replaced by its image under α. We describe the two transcosets as "copies". Discard σ and go to (1).

(5) Compute $\nu(A)$.

(6) Compute and print τ using the relationship:

$$(at)\tau = a\pi(t)\ t\nu(a), \text{ for all } a \in A,\ t \in T.$$

This yields τ as the product of disjoint cycles on the elements of AT.

(7) Go to (1) until $Sym_1(A)$ exhausted, whereupon stop.

Whenever σ was discarded, a message was printed indicating the particular permutation σ' earlier in the enumeration which yielded an identical transcoset, or a copy. Thus any given permutation in $Sym_1(A)$ can be traced back to one of the permutations defining an elementary transcoset appearing in the output.

3.2.2. Enumerating the elements of the Symmetric Group.

We shall now describe the method used to enumerate the elements of the symmetric group, that is, to place the permutations forming Sym(n) into 1-1 correspondence with the natural numbers, 0,1,2,...(n!-1).

Several methods of generating successively all the permutations of the symmetric group, Sym(n), have been published, and compared for speed of computer execution. See for example R.J. Ord-Smith(1970). The method used here is the author's own, although similar principles may be discerned in a number of other methods. For example, the principle of mixed-radix arithmetic used in association with permutations is apparently not novel; being attributed (without reference) to M. Hall, jnr. by Ord-Smith (ibid).

The originality of the present method, therefore, lies not so much in the conversion of a natural number into a unique permutation, but rather in the rapid execution of the reverse process, by which steps (3) and (4) of algorithm 3.2.1.1. may be performed without the need to store all permutations of $Sym_1(A)$, nor to search them.

3.2.2.1. DEFINITION. The symmetric group, Sym(n), will be assumed to be defined on the numerals: $1,2,3,\ldots n,n+1,\ldots$ ad ∞, of which the numerals, $n+1$, $n+2$, ... ad ∞ are considered to be fixed.

We shall assign to each permutation of Sym(n) a unique number, with the property of being independent of the value of n. Under the above definition, a permutation, $\sigma \in$ Sym(n) lies also in Sym(n') for all $n' \geq n$. The numerical representation, m_σ, as we shall call it, will be independent of the particular Sym(n') assumed to contain it.

3.2.2.2. PROPOSITION. Let c_r be the cycle, $(1,2, \ldots,r)$, $2 \leq r \leq n$. Any given permutation, σ, in Sym(n) is uniquely expressible in the form:

$$\sigma = c_n^{i_n} c_{n-1}^{i_{n-1}} \ldots\ldots c_2^{i_2}.$$

where $r > i_r \geqslant 0$.

PROOF. By induction on n.

The proposition is true for $n = 2$, whereupon i_2 may take the values 0 or 1.

Now Sym(n) is expressible as a Zappa product:

$\mathrm{Sym}(n) = C_n \, \mathrm{Sym}_n(n)$, where $\mathrm{Sym}_n(n)$ is the stabiliser of n in Sym(n). By definition 3.2.2.1., $\mathrm{Sym}_n(n) = \mathrm{Sym}(n-1)$.

Hence every σ Sym(n) has the unique expression:

$$\sigma = c_n^{i_n} \sigma', \text{ for unique } c_n^{i_n} \in C_n, \text{ and } \sigma' \in \mathrm{Sym}(n-1).$$

$$= c_n^{i_n} c_{n-1}^{i_{n-1}} \;\ldots\ldots\; c_2^{i_2},$$

which is true by assumption of the truth of the proposition for (n-1).

3.2.2.3. DEFINITION. An <u>ascending radix numeral</u> is a finite string of digits (shown enclosed by quotes,"), where the rightmost digit has radix 2, the radix of each successive digit to the left increasing by 1.

Two such strings will be equated if they differ only by preceding zero digits.

3.2.2.4. EXAMPLE. "5,4,3,2,1" is a valid ascending radix numeral.

The natural lexicographic ordering of ascending radix numerals places them in 1-1 correspondence with the natural numbers, thus:

"0" $\longleftrightarrow$ 0;

"1" $\longleftrightarrow$ 1;

"5,4,3,2,1" $\longleftrightarrow$ 719;

"1,0,0,0,0,0" $\leftrightarrow$ 720 = 5!.

3.2.2.5. PROPOSITION. Let $\sigma \in \mathrm{Sym}(n)$ have the unique expression: $c_n^{i_n} c_{n-1}^{i_{n-1}} \cdots\cdots c_2^{i_2}$. Then "$i_n$, i_{n-1}, ..., i_2" is a valid ascending radix numeral and represents a unique number, m_σ, called the <u>numerical representation</u> of σ.

PROOF. By its position in the string, the radix of i_r is r, $n \geqslant r \geqslant 2$. By the assumptions of proposition 3.2.2.2., $r > i_r \geqslant 0$. Hence "i_n, i_{n-1}, ..., i_2" is a valid ascending radix numeral.

Since by definition, "0, 0, ..., 0, i_n, i_{n-1}, ..., i_2" is equated to "i_n, i_{n-1}, ..., i_2", m_σ is independent of the value of n in whatever Sym(n) is supposed to contain σ. Hence m_σ corresponds uniquely to σ.

To convert any finite natural number, m_σ, into the corresponding permutation, σ, the ascending radix numeral equated with m_σ is found, from which follows the unique sequence of elements of C_r, $r = 2,3,\ldots,n$, to define σ. n is found as the radix of the leftmost non-zero digit in the ascending radix numeral of m_σ.

To convert a given permutation, σ , into its numerical representation, m_σ, a rapid ALGOL procedure was developed, behaving essentially like the following manual algorithm:

3.2.2.6. ALGORITHM. To obtain the unique numerical representation of a permutation, σ.

(1) Write down σ in cyclic form, followed by a string

of ... c_r ...; thus:

$$(\sigma^{-1})\ c_n\quad c_{n-1}\quad c_{n-2}\quad \ldots\ldots\quad c_3\quad c_2,$$

where n is the highest value point moved under σ.

(2) For r = n, n-1, ..., 3, 2 successively:

choose i_r such that $(\sigma^{-1})\ c_n^{i_n}\ c_{n-1}^{i_{n-1}}\ \ldots\ c_r^{i_r}$ fixes r.

(3) Convert "i_n, i_{n-1}, ..., i_2" to a natural number, m_σ.

NOTES. (i) Step (2) is easy to perform without writing down the permutation forms of $c_j^{i_j}$ explicitly, $n \geqslant j \geqslant 2$, since the image of point s under $c_j^{i_j}$ is simply:

$$\begin{array}{ll} s+i_j \pmod j), & \text{if } s \leqslant j; \\ s & \text{if } s > j. \end{array}$$

(ii) No successive application of step (2) can move a point, $s > r$, once fixed. Thus the algorithm terminates with all points fixed, i.e. with:

$$(\sigma^{-1})\ c_n^{i_n}\ c_{n-1}^{i_{n-1}}\ \ldots\ldots\ c_2^{i_2} = (1).$$

3.2.2.7. EXAMPLE. Obtain the numerical representation of (1,4,2)(3,5,6,7)

Step(1): $(1,2,4)(3,7,6,5)\ c_7\quad c_6\quad c_5\quad c_4\quad c_3\quad c_2$

Step (2): r = 7. Since $\sigma^{-1}:7\to 6$ we need $c_7^{i_7}:6\to 7$, i.e. $i_7 = 1$.

r = 6. " $\sigma^{-1}c_7^1:6\to 6$ " $c_6^{i_6}:6\to 6$ " $i_6 = 0$.

r = 5. " $\sigma^{-1}c_7^1c_6^0:5\to 4$ " $c_5^{i_5}:4\to 5$ " $i_5 = 1$.

r = 4. " $\sigma^{-1}c_7^1c_6^0c_5^1:4\to 3$ " $c_4^{i_4}:3\to 4$, " $i_4 = 1$.

r = 3. $\sigma^{-1}c_7^1c_6^0c_5^1c_4^1:3\to 3$, $i_3 = 0$.

r = 2. $\sigma^{-1}c_7^1c_6^0c_5^1c_4^1c_3^0:2\to 2$, $i_2 = 0$.

Step (3): The ascending radix numeral corresponding to $(1,4,2)(3,5,6,7)$ is "1,0,1,1,0,0".

$$m_\sigma = 1.6! + 1.4! + 1.3! = 750.$$

3.2.3. <u>The fully collapsed elementary transcosets of D_8</u>.

Table 1 lists salient details of the fully collapsed elementary transcosets of $A = D_8$, the dihedral group of order 8, with all copies suppressed.

There are 137 distinct non-trivial fully collapsed elementary transcosets of D_8, of which 68 lie in $AB \cong Alt(8)$, the alternating group on 8 points. The latter property is determined by whether σ, defining the elementary transcoset by 2.2.1.8., is an odd or even parity permutation. Since $\lambda(A)$ here consists of even parity permutations only, the set of permutations, $\pi(T) \ni \sigma$, for each defined elementary transcoset, $\{A,T,\tau\}$, are all of the same parity, i.e. either all odd, or all even. This yields the result:

3.2.3.1. PROPOSITION. $Alt(A)$ is factorisable into $\lambda(A)$ and $Alt_1(A)$, the stabilizer of element 1 in $Alt(A)$, if and only if for all $a \in A$, $\lambda(a)$ is of even parity, i.e. if $|A|/2$ is not an odd integer.

The parity of a permutation can be found from the numerical representation as follows:

3.2.3.2. PROPOSITION. The parity of a permutation is equal to the numerical parity of the sum of alternate digits, starting with and including the rightmost, in the ascending radix form of its numerical representation.

PROOF. Every alternate digit from and including the rightmost in the ascending radix numerical representation of a given permutation, σ, represents the exponent of a cyclic element of even length (hence of odd parity) in an expansion of σ. Hence the sum of these digits denotes the number of odd parity factors in a product of permutations equal to σ. If this number is odd the parity of σ is odd, otherwise it is even.

3.2.3.3. EXAMPLE. 787 = "$1,\underline{0},2,\underline{3},0,\underline{1}$" is the numerical representation of the permutation: (1,3,4,2,5,6,7). The sum of alternate digits: x,0,x,3,x,1, is 4, which is even.

In computing table 1, the following representation, $\lambda(A)$, of D_8 was used, shown together with its automorphism group, Aut(A):

$\lambda(0) = (1)$,

$\lambda(1) = (0,1,2,3)(4,5,6,7)$,

$\lambda(2) = (0,2)(1,3)(4,6)(5,7)$,

$\lambda(3) = (0,3,2,1)(4,7,6,5)$,

$\lambda(4) = (0,4)(1,7)(2,6)(3,5)$,

$\lambda(5) = (0,5)(1,4)(2,7)(3,6)$,

$\lambda(6) = (0,6)(1,5)(2,4)(3,7)$,

$\lambda(7) = (0,7)(1,6)(2,5)(3,4)$;

$\mathrm{Aut}(A) = \{(1),\ (4,5,6,7),\ (4,6)(5,7),\ (4,7,6,5),$
$(1,3)(4,5)(6,7),\ (1,3)(4,6),\ (1,3)(5,7),$

$(1,3)(4,7)(5,6)\}$.

The computer programme output τ for each new elementary transcoset, $\{A,T,\tau\}$, discovered. For space considerations, the entire transcoset will not be exhibited here in each case, but instead only the distinct lengths of the cycles of τ. By proposition 3.1.1.2, these values are divisors of the orders of elements contained in AT embedded in some host group.

Elementary transcosets are indexed by the numerical representation of $\sigma = \pi(t)$ for some $t \in T$ which defines them, and will be referred to by using this number as a label.

The elementary transcosets: 147, 240 and 830 lie in transcosets of A defining groups isomorphic to S_{168}. We exhibit one of these transcosets as the generalised transcoset of A and B with which it is associated (B being non-abelian of order 21), namely the one containing 147. It has been assembled from 147 and copies of 240 and 830.

3.2.3.4. EXAMPLE. A transcoset of $A = D_8$ defining the simple group S_{168}.

A is the set $\{0,1,2,3,4,5,6,7\}$; 0 is the identity element, thus $E = \{0\}$; B is supplied as the union of E and a set of 20 indeterminates:

b,c,d,e,f,g,h,i,j,k,m,n,p,q,r,s,t,u,v,w.

The transcoset is presented below as a tabulation of $(AB)\chi$.

The location indexed by $a \in A$ and $b \in B$ contains $(ab)\chi = a'b'$, for some $a' \in A$, $b' \in B$. Thus $(1b)\chi = b1 = 2c$; $(1c)\chi = c1 = 6d$; etc. Also, for all $b \in B$, $0b = (0b)\chi = b$.
Alongside each element of set B, its representation as provided by $\pi(B)$ is displayed, as determined by the relevant elementary transcoset brought in.

	(1)	T_1	0	1	2	3	4	5	6	7
147:	(1,2,4)(3,5,6)	T_2	b	2c	4d	5e	1f	6g	3h	7i
	(1,6,5)(3,2,7)		c	6d	7e	2b	4g	1h	5i	3f
	(1,3,6,7,2,4,5)		d	3e	4b	6c	5h	1i	7f	2g
	(1,5,4,2,7,6,3)		e	5b	7c	1d	2i	4f	3g	6h
	(1,6,7)(3,5,4)		f	6i	2h	5g	3b	4e	7d	1c
	(1,5,6)(3,7,2)		g	5f	3i	7h	4c	6b	1e	2d
	(1,7,6)(3,4,5)		h	7g	2f	4i	5d	3c	1b	6e
	(1,4,2)(3,6,5)		i	4h	1g	6f	2e	3d	5c	7b
≡ 830:	(2,6,3)(4,5,7)	T_3	j	1k	6m	2n	5j	7n	3m	4k
	(1,5,2)(4,6,7)		k	5m	1n	3j	6n	2m	7k	4j
	(1,4,7,3,5,2,6)		m	4n	6j	5k	7m	2k	1j	3n
	(1,2,3,4,6,5,7)		n	2j	3k	4m	6k	7j	5n	1m
≡ 830:	(1,2,5)(4,7,6)	T_4	p	2s	5r	3q	7q	1r	4s	6p
	(2,3,6)(4,7,5)		q	1p	3s	6r	7p	4q	2r	5s
	(1,6,2,5,3,7,4)		r	6q	5p	7s	1s	3p	2q	4r
	(1,7,5,6,4,3,2)		s	7r	1q	2p	3r	6s	4p	5q
≡ 240:	(1,7,3)(4,6,2)	T_5	t	7u	4t	1u	6t	5u	2t	3u
	(1,3,7)(4,2,6)		u	3t	6u	7t	2u	5t	4u	1t
≡ 240:	(1,3,4)(5,2,7)	T_6	v	3w	7v	4w	1w	2v	6w	5v
	(1,4,3)(5,7,2)		w	4v	5w	1v	3v	7w	6v	2w

We assemble the following elementary transcosets of $A = D_8$.

(1): The trivial transcoset, $\{A,E,(1)\}$, of A itself.

(2): $\{A,T_2,\tau\}$, identical to transcoset 147.

(3): $\{A,T_3,\tau\}$, a copy of transcoset 830. The required copy is obtained by conjugating τ by $\bar{\alpha}$, where $\bar{\alpha}: AT_3 \rightarrow AT_3$ is related to the automorphism, $\alpha = (4,6)(5,7)$ of A as follows:

$$\bar{\alpha}: at \rightarrow (a\alpha)t, \text{ for all } a \in A,\ t \in T_3.$$

α is chosen to yield $\pi(T) \in gp\{\pi(T_2)\}$.

(4): A second copy, $\{A,T_4,\tau\}$, of transcoset 830. AT_4 is the inverse double coset to AT_3. In fact $T_4 = T_3^{-1}$.

α here is chosen as $(1,3)(4,5)(6,7)$.

(5): A copy, $\{A,T_5,\tau\}$, of transcoset 240. α here is chosen as $(1,3)(5,7)$. AT_5 is a self-inverse double coset.

(6): A second copy, $\{A,T_6,\tau\}$, of transcoset 240. α here is chosen as $(7,6,5,4)$. AT_6 is likewise self-inverse.

The concatenation of transcosets (1)-(6) defines a group (a Zappa product) AB, by theorem 3.1.3.1., because $\pi(B)$ is a group of distinct permutations. $\pi(B)$ faithfully represents B (by anti-isomorphism), thus defining a group over the previously structureless set B.

The simplicity of this Zappa product is immediately apparent from corollary 3.1.3.3., because A and B are of co-prime order, and neither A nor B possess proper normal subgroups $L \triangleleft A$, $M \triangleleft B$, such that $(LB)\chi = LB$, or $(AM)\chi = AM$.

CHAPTER 4. On a Method of constructing Generalised Transcosets.

Chapters 1 and 2 have been devoted to developing a theory of constructing all finite groups generated by two fully-inconjugate subgroups, A and B. This problem has been reduced to that of simultaneously forming a pair of associated transcosets, $\{A,(SB),\chi\}$, $\{B,(SA),\chi^{-1}\}$, of the generalised transcoset, $\{A,S,B,\chi\}$ by concatenating basic units called elementary transcosets, of A and B respectively.

In the case of A and B prime-cyclic, the number of distinct, non-equivalent elementary transcosets to be chosen from is small. In the case of A, say, they are either of type H = E (of which there is only one), or of type H = A (corresponding to trivial double cosets).

Although the scope of the present work has been viewed as laying the foundations for such methods of constructing finite groups by non-normal extension, it would not be complete without some simple example serving to indicate how such methods might be developed.

Accordingly we pursue in this chapter the non-trivial case of lowest order, namely where A and B are both of order 2, as an elementary application of transcoset theory. In passing we shall furnish methods whereby examples of non-group generalised transcosets such as those exhibited in Chapter 1 may be constructed.

4.1. The case of A and B both of order 2.

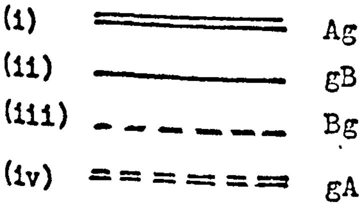

Fig. 1.

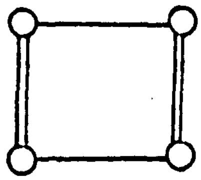

Fig. 2.
AgB unit.

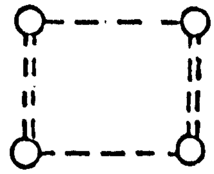

Fig. 3.
BgA unit.

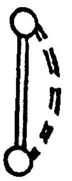

Fig. 4.

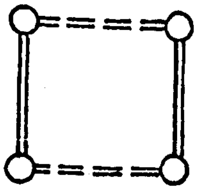

Fig. 5.

AgA unit.

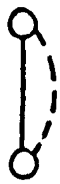

Fig. 6.

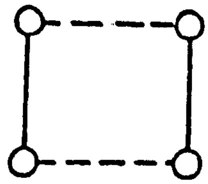

Fig. 7.

BgB unit.

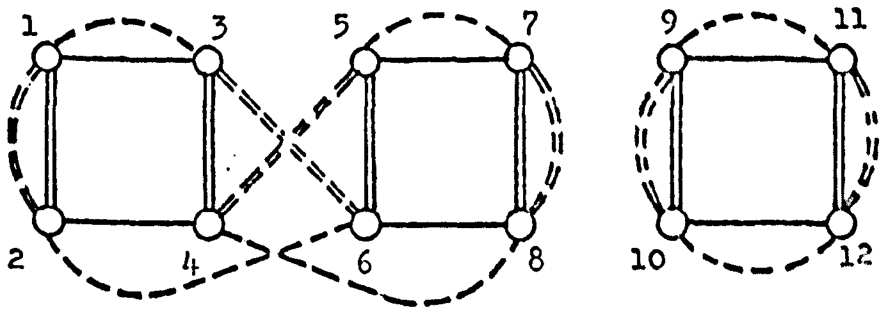

Fig. 8. Examples 1.3.3.1. & 1.3.3.5.

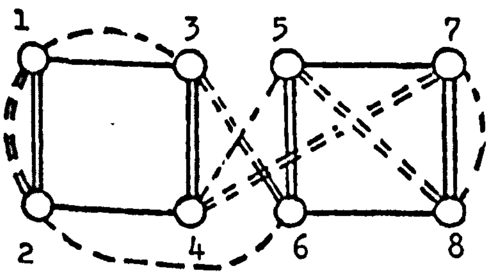

Fig. 9. Example 1.3.3.3.

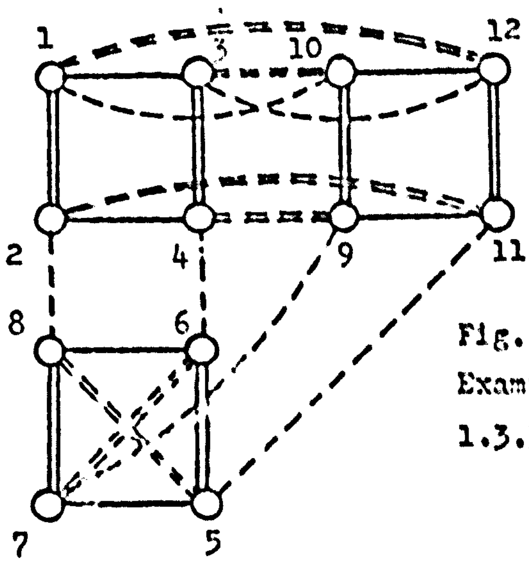

Fig. 10.
Example
1.3.3.4.

4.1.1. A schematic method of representing the generalised transcoset of A and B.

We introduce a schematic method of constructing all generalised transcosets of A and B, which readily handles the case of A and B both of order 2.

We shall use it to survey all generalised transcosets of A and B, exhibiting a complete description of $\{A,S,B,\chi\}$ for the case $|S| = 1$, and also for the case where $\{A,S,B,\chi\}$ defines a group.

Let us represent the elements of ASB by the nodes of a (topological) graph, with edges each of which is assigned one of four colours. Two nodes are joined by an edge of the appropriate colour according to whether they lie in the same right or left coset of A or B in the set ASB.

4.1.1.1. DEFINITION. Two elements, asb, $a's'b' \in ASB$, the formal product of A, S and B, lie in the same:

(i) right coset of A, Ag, if $asb = (a''a')s'b'$;

(ii) left coset of B, gB, if $asb = a's'(b'b'')$;

(iii) right coset of B, Bg, if $(asb)\chi = (a's'(b''b'))\chi$;

(iv) left coset of A, gA, if $(asb)\chi = ((a'a'')s'b')\chi$;

for some $a'' \in A$, or some $b'' \in B$. See Fig. 1 for the appropriate coloured edge.

4.1.1.2. PROPOSITION. A graph represents some ordered quadruple, $\{A,S,B,\chi\}$, where A and B are both of order 2, if and only if the following conditions are satisfied:

(i) The number of nodes is divisible by 4.

(ii) Every node meets precisely one edge in each of the four colours.

(iii) Every node lies in one each of the subgraphs, Fig. 2, Fig. 3, corresponding to double cosets of the form AgB, BgA, $g \in ASB$.

PROOF. Given an ordered quadruple, $\{A,S,B,\chi\}$, where A and B are of order 2, assign nodes in 1-1 correspondence with the elements of set ASB. This satisfies condition (i). Join two nodes with an edge of the appropriate colour according to whether they lie in the same right or left coset of A or B.
Every element must lie in precisely one left (right) coset of A (B). This satisfies condition (ii).
Every $g \in ASB$ lies in a subset, AsB, also in a subset, $(As'B)\chi$, for some $s,s' \in S$. The nodes corresponding to AsB must be joined as in Fig. 2, and the nodes corresponding to $(As'B)\chi$ must be joined as in Fig. 3. Hence condition (iii) is satisfied.
Conversely, given a graph satisfying (i)-(iii), then the nodes may be partitioned into r disjoint subsets of 4 nodes, forming subgraphs of the type of Fig. 2. They are disjoint because no node can lie in two distinct subgraphs (Fig. 2) without meeting more than one edge of the same colour.
Now each set of 4 nodes can be placed in 1-1 correspondence with the elements of the formal product, AsB, for some pair of order 2 groups, A and B, and some $s \in S$, a set of r indeterminates, so as to satisfy definition 4.1.1.1.

The foregoing paragraph holds true reading "Fig. 3" for "Fig 2", and "$(ASB)\chi$" for "ASB". Thus we have defined the formal product, ASB, and its image under χ, and hence the ordered quadruple, $\{A,S,B,\chi\}$.

We aim to show that each node in a given graph satisfying 4.1.1.2. (i)-(iii) lies also in subgraphs of the types shown in Figs. 4 or 5, 6 or 7, if and only if the graph represents a generalised transcoset of A and B. We shall proceed towards this goal by a sequence of lemmas.

4.1.1.3. LEMMA. An ordered triple, $\{A,T,\tau\}$, of $A = C_2$ (not necessarily in standard form) is an elementary transcoset of A if and only if either:

(i) $|T| = 1$, or:

(ii) $|T| = 2$ and $(At)\tau$ equals neither At nor At', where $T = \{t,t'\}$.

PROOF. If A,T, is an elementary transcoset then $|T| = (A:H_t) = 1$ or 2. If $|T| = 2$ and $(At)\tau = At'$, t, $t' \in T$, then $(At')\tau = At$, and $\{A,T,\tau\}$ is identical to the concatenation of two elementary transcosets.

Conversely if $|T| = 1$ then $\tau = (1)$ or (1t, at) only. In either case, $\lambda(a) = \rho(a) = (1t,\ at)$ for $a \neq 1$, hence the A-commuting condition is satisfied.

If $|T| = 2$ then let $T = \{t,\ u\}$. Then for $a \neq 1$:

$$\lambda(a) = (1t,\ at)(1u,\ au);$$

$$\rho(a) = (1t,\ 1u)(at,\ au),\ \text{or}\ (1t,\ au)(at,\ 1u),$$

for any given value of τ, for example (at, au). That this yields a transcoset follows from the fact that $\lambda(a)$ commutes with either of the two possible values of $\rho(a)$, which implies the A-commuting condition.

4.1.1.4. LEMMA. If an elementary transcoset, $\{A,T,\tau\}$, is embedded in a generalised transcoset of A and B, then in the graph representing the generalised transcoset, $\{A,S,B,\chi\}$, the set AT corresponds to a subgraph of the form of Fig. 4 or 5. Conversely if the graph representing an ordered quadruple, $\{A,S,B,\chi\}$, contains a subgraph of the form of Fig. 4 or 5 then this corresponds to an embedded elementary transcoset of A.

PROOF. If $|T| = 1$ then the elements, 1t, at, lie in the same left and right coset of A. Hence the corresponding nodes are joined as in Fig. 4.

If $|T| = 2$ then the four elements form two right cosets of A, say At, Au, and two distinct left cosets, $(At)\tau$, $(Au)\tau$, neither of which equals At. Fig. 5 then illustrates all ways of achieving this.

Conversely suppose a subgraph of the form of Fig. 4 or 5 occurs in the graph of an ordered quadruple, $\{A,S,B,\chi\}$. Bring the associated ordered triple, $\{A,(SB),\chi\}$, to standard form, say $\{A,R,\chi'\}$, by an equivalence-transformation as described in Chapter 2. Then $R \supseteq T$ such that set AT corresponds to the nodes of the subgraph, since AT forms entire right cosets of A. If Fig. 4 applies to the subgraph then $|T| = 1$ and τ fixes AT, whence the lemma follows by 4.1.1.3. If Fig. 5 applies then $|T| = 2$, τ fixes AT and a left coset of A equals no right coset of A in AT. Hence again the lemma follows by 4.1.1.3.

A similar proposition holds for B, with Figs. 6 and 7.

4.1.1.5. PROPOSITION. $\{A,S,B,\chi\}$ is a generalised transcoset of A and B, both of order 2, if and only if every node of the graph which represents it lies in a subgraph of the form of either Fig. 4 or 5, and also in a subgraph of the form of either Figs. 6 or 7.

PROOF. These subgraphs (Fig. 4 or 5) must be disjoint, otherwise a node meets more than one edge of the same colour. Similarly for Figs. 6 and 7. $\{A,S,B,\chi\}$ is a generalised transcoset if and only if it satisfies the A- and B-commuting conditions, therefore if and only if every element of ASB lies in a pair of embedded elementary transcosets of A and of B respectively. Hence the proposition follows by lemma 4.1.1.4.

4.1.1.6. DEFINITION. Two generalised transcosets of A and B are equivalent if and only if their respective associated transcosets of A and of B are equivalent, i.e. if and only if they share the same derived groups, $\lambda(A)$, $\rho(A)$, $\lambda(B)$, $\rho(B)$.

4.1.1.7. PROPOSITION. Two generalised transcosets are equivalent if and only if they share the same graph.

PROOF. Form 2-cycles from each pair of elements corresponding to a pair of nodes joined by an edge of a given colour. This yields: ($a \neq 1$, and the roman numerals corresponding to those of definition 4.1.1.1.)

(i): $\lambda(a)$; (ii): $\rho(b)$; (iii): $\lambda(b)$; (iv): $\rho(a)$.

This yields a unique set of derived groups, $\lambda(A)$, $\rho(B)$, $\lambda(B)$, $\rho(A)$, from the given shared graph. Hence all generalised transcosets with this graph are equivalent.

Conversely any two equivalent generalised transcosets of A and B share the same derived groups, of which the cycles of $\lambda(a)$, $\rho(b)$, $\lambda(b)$, $\rho(a)$, define left/right cosets of A and B, hence edges of corresponding colours. This defines the same graph for both generalised transcosets.

Thus we effectively construct all generalised transcosets of A and B, unique to equivalence, by constructing all non-isomorphic graphs satisfying the conditions of both propositions 4.1.1.2., and 4.1.1.5.

Since there are several infinite series of such graphs, we shall restrict our attention to two special cases, namely to the graphs of $\{A,S,B,\chi\}$ where:

- $|S| = 1$, (see below)
- a group is defined, (see subsection 4.1.2.).

For the sake of interest we exhibit the graphs of examples 1.3.3.1. and 1.3.3.5. (Fig.8); 1.3.3.3. (Fig. 9); 1.3.3.4. (Fig. 10); of Chapter 1.

Note that the generalised transcosets, 1.3.3.1. and 1.3.3.5. are equivalent, as was mentioned in Chapter 1, and that 1.3.3.3. was there shown not to be a generalised transcoset.

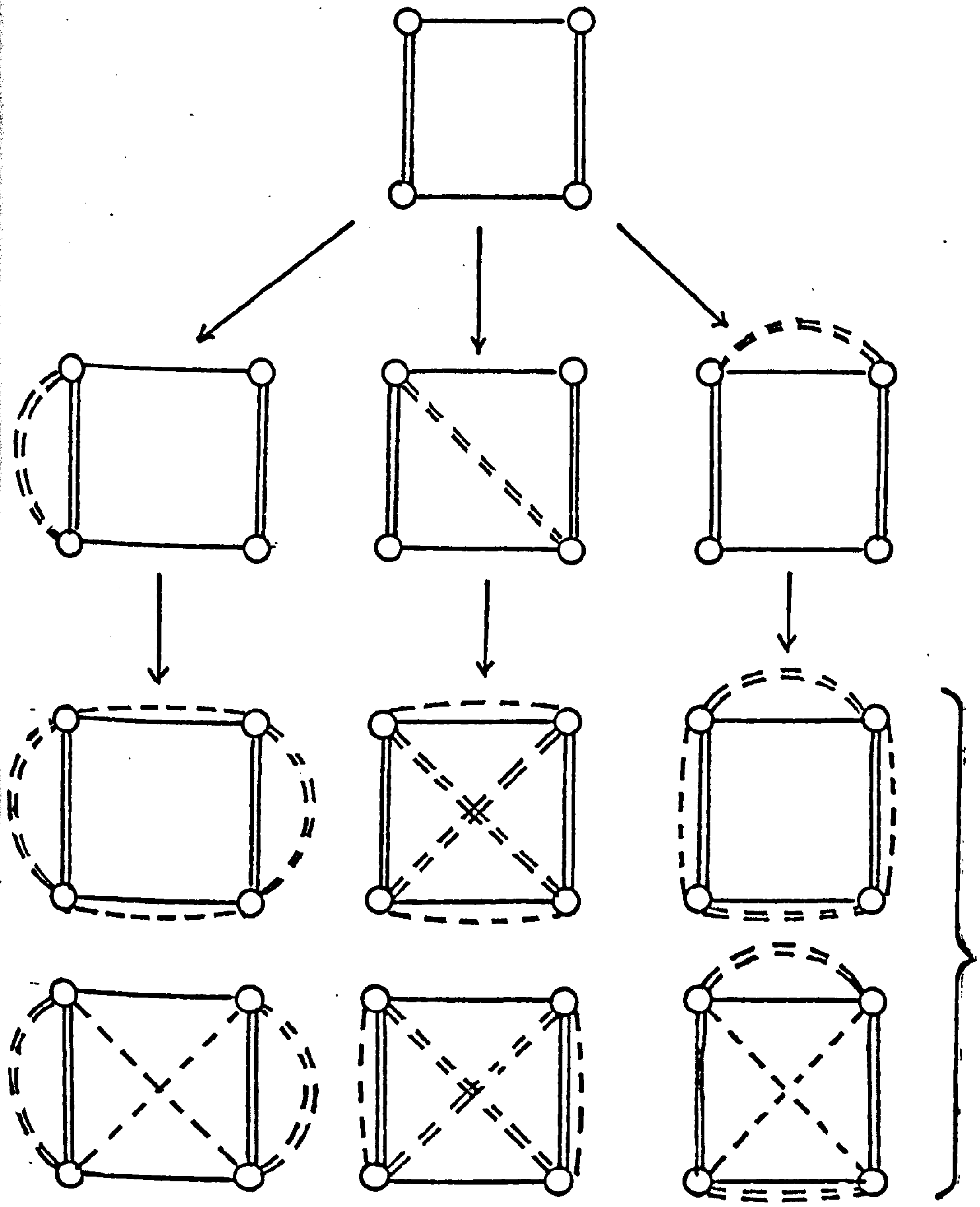

Fig. 11.

The six distinct generalised transcoset graphs for the case $|S| = 1$.

Fig. 11 shows a hierarchy of subgraphs, terminating at the third (the lowest) level in the distinct non-isomorphic graphs of generalised transcosets, $\{A,S,B,\chi\}$, where $|S| = 1$.

At the first level we start with the subgraph (Fig. 2) which all such graphs must contain.

The second level illustrates the three distinct ways of adding an edge of colour (iv). The remaining two nodes can only be joined similarly.

The third level represents all six distinct ways of adding a pair of edges of colour (iii) to complete the BgA type subgraph, (mandatory), of Fig. 3. The resulting six distinct graphs each satisfy the conditions for representing generalised transcosets.

4.1.2. <u>The generalised transcosets of A and B, both of order 2, which define groups.</u>

We shall construct a graph, by stages, which represents all such groups. It must consist of a finite number, r, of sets of four nodes each, each forming a disjoint subgraph of the type shown in Fig. 2, called an AgB unit.

Let one of the nodes correspond to the identity, $1 \in A \cap B$. Since χ, of the represented generalised transcoset, must fix $A11 \cup 11B$ then 1 must lie in the subgraph shown in Fig. 12.

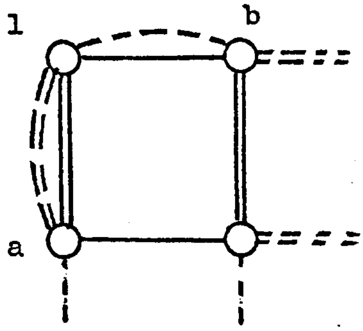

Fig. 12.

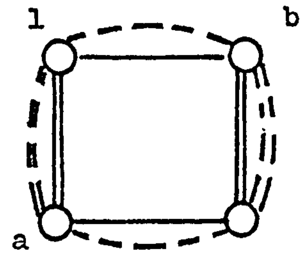

Fig. 13.

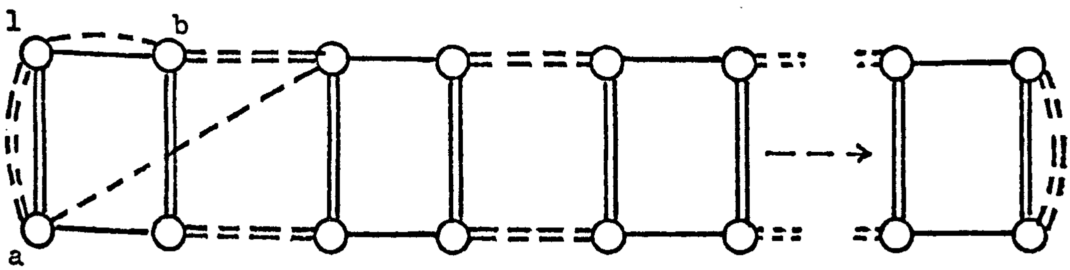

Fig. 14.

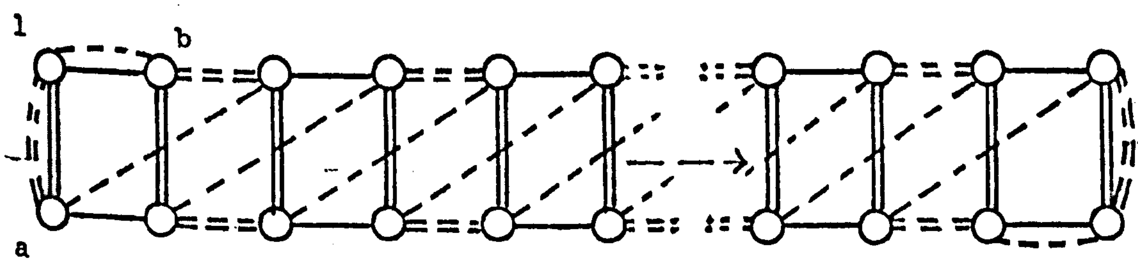

Fig. 15.

The series of graphs of generalised transcosets which define groups.

The two unassigned colour (iv) edges may either be the same edge, or attached to another AgB unit so as to form an AgA unit (Fig. 5). If the same edge, the Fig. 13 results. This defines a subset AS'B of ASB violating condition 1.3.1.4. (i). Hence any larger generalised transcoset containing this cannot define a group.

Otherwise if attached to another AgB unit a finite chain of r such units is formed, $r > 1$. See Fig. 14. The diagonal colour (iii) edge is mandatory to complete the BgA unit containing node 1. Next, consecutive diagonals are mandatory to complete successive BgB, BgA units (Fig. 15).

The final two nodes must be joined by a colour (iii) edge as shown to complete the final BgA subgraph. No more AgB units can be added without violating 1.3.1.4. (i) as above. Hence we have shown that there exists only one graph to isomorphism, representing generalised transcosets of A and B, both of order 2, defining groups.

The groups thus defined are all dihedral, as may be shown by the following theorem: (See D. Gorenstein (1968))

4.1.2.1. THEOREM. If x and y are involutions (i.e. order 2 elements) of group G then both x and y invert the product, xy, and so gp $\{x, y\}$ is dihedral.

4.2. Conclusion.

In this chapter we have introduced a method of finding all generalised transcosets of A and B for the fairly straightforward case of A, B both of order 2. The schematic method used suggests generalisation to higher order, prime-cyclic groups, A and B, by use of directed coloured edges. Such a method does not readily lend itself to manual handling, but may allow computer searches of groups of a particular order generated by two prime-order elements, or even for non-group generalised transcosets.

Although we have only offered easy examples in this chapter, these have been included as suggestive applications of the theory of transcosets developed in Chapters 1 and 2, which has been the chief purpose of this thesis.

--ooOoo--

TABLE 1. The fully collapsed elementary transcosets of D_8.

n.r.	corresp. perm.	T	cycle-lengths	Alt(A)?	Notes.
1	(1,2)	8	2,3,4.		
2	(1,2,3)	8	2,3,4,6.	Alt	
5	(1,3)	4	1,2.		$= \emptyset_A$
6	(1,2,3,4)	8	2,3,4,5,6,8.		
7	(2,3,4)	8	2,3,5,6,7.	Alt	
8	(1,3,4,2)	8	2,3,4,5,8.		
9	(1,3,4)	8	2,3,6.	Alt	
10	(3,4)	4	1,2,4,8.		
11	(1,2)(3,4)	8	2,4,6,7.	Alt	
12	(1,3)(2,4)	8	2,4,6.	Alt	
13	(1,3,2,4)	8	2,3,4,5,8.		
24	(1,2,3,4,5)	4	2,3,5.	Alt	
25	(2,3,4,5)	8	2,3,4,6,8.		
26	(1,3,4,5,2)	8	2,3,4,5,6,7.	Alt	
27	(1,3,4,5)	8	2,3,4,5.		
29	(1,2)(3,4,5)	8	2,3,4,5,6,8.		
30	(1,3)(2,4,5)	8	2,3,5,6,8.		
31	(1,3,2,4,5)	8	2,3,4,5,6,7.	Alt	
32	(2,4,5,3)	4	1,2,3,4,5.		
33	(1,2,4,5,3)	8	2,3,4,5,6,7.	Alt	
36	(1,4,5,3,2)	8	2,3,5,6,7.	Alt	
40	(1,4,5,2,3)	8	2,3,4,5,6,7.	Alt	
41	(1,4,5)(2,3)	8	2,3,4,5,6,8,		
43	(1,2)(4,5)	4	1,2,3,4,6.	Alt	
44	(1,2,3)(4,5)	4	2,3,4,8.		
47	(1,3)(4,5)	8	2,3,4,6.	Alt	
48	(1,3,5,2,4)	8	2,3,5,6,7.	Alt	
49	(1,3,5)(2,4)	4	2,3,4,8.		
50	(2,4)(3,5)	4	1,2,4.	Alt	
51	(1,2,4)(3,5)	8	2,3,4,5,6,8.		
52	(1,2,4,3,5)	8	2,3,4,5,6,7.	Alt	
53	(2,4,3,5)	8	2,4,5,8.		
54	(1,4,2)(3,5)	8	2,3,4,5,6,8.		
55	(1,4)(3,5)	8	2,3,4,5,6.	Alt	

TABLE 1. (continued)

n.r.	corresp. perm.	T	cycle-lengths	Alt(A)?	Notes
56	(1,4,3,5)	8	2,3,4,5,8.		
57	(1,4,3,5,2)	8	2,3,4,5,6,7.	Alt	
90	(1,3,4)(2,5)	4	2,3,4,8.		
91	(1,3,4,2,5)	8	2,3,4,5,6,7.	Alt	
93	(1,2,5,3,4)	8	2,3,4,5,6,7.	Alt	
94	(1,2,5)(3,4)	8	2,3,5,8.		
95	(2,5)(4,3)	2	1,2,6	Alt	
108	(1,5,2,3,4)	8	2,3,4,5,6,7.	Alt	
109	(1,5)(2,3,4)	8	2,3,5,8.		
122	(1,3,4,5,6,2)	8	2,3,4,5,6,8.		
123	(1,3,4,5,6)	8	2,3,4,5,6.	Alt	
126	(1,3)(2,4,5,6)	8	2,3,4,6,7.	Alt	
127	(1,3,2,4,5,6)	8	2,3,4,5,6,8.		
132	(1,4,5,6,3,2)	4	1,3,6,8.		
142	(1,3,2)(4,5,6)	4	1,2,3,4.	Alt	
143	(1,3)(4,5,6)	8	2,3,4,8.		
144	(1,3,5,6,2,4)	8	2,3,4,6.		
145	(1,3,5,6)(2,4)	8	2,3,4,5,6,7.	Alt	
147	(1,2,4)(3,5,6)	8	2,3,4,7.	Alt	$\subset S_{168}$
148	(1,2,4,3,5,6)	8	2,3,4,6,8.		
150	(1,4,2)(3,5,6)	4	1,3,7.	Alt	
151	(1,4)(3,5,6)	8	2,3,4,8.		
152	(1,4,3,5,6)	8	2,3,4,5,6,7.	Alt	
153	(1,4,3,5,6,2)	4	2,3,4,5,6,8.		
154	(1,4)(3,5,6,2)	4	2,3,4,6.	Alt	
155	(1,4,2,3,5,6)	4	2,3,4,5,6,8.		
157	(1,2)(3,5,6,4)	8	2,3,4,5,7.	Alt	
160	(1,3,5,6,4,2)	4	1,2,3,4,5.		
161	(1,3,5,6,4)	8	2,3,4,5,6,7.	Alt	
168	(1,4,2,5,6,3)	8	2,5,6,8.		
169	(1,4)(2,5,6,3)	8	2,3,4,5,6,7.	Alt	
170	(1,4,3,2,5,6)	8	2,3,5,6,8.		
175	(1,2,5,6,4,3)	4	1,2,4,6,8.		
178	(1,3)(2,5,6,4)	8	2,3,4,5,6,7.	Alt	
179	(1,3,2,5,6,4)	8	2,5,6,8.		
186	(1,3,4)(2,5,6)	8	2,3,4,6,7.	Alt	
189	(1,2,5,6,3,4)	8	2,6,8.		

TABLE 1. (continued)

n.r.	corresp. perm.	T	cycle-lengths	Alt(A)?	Notes
196	(1,5,6,4,2,3)	8	2,3,4,6,8.		
197	(1,5,6,4)(2,3)	4	2,4,6,7.	Alt	
209	(1,5,6,2)(4,3)	4	1,2,4,7.	Alt	
211	(1,5,6,3,2,4)	8	2,3,4,6,8.		
212	(1,5,6)(2,4,3)	8	2,3,4,6,7	Alt	
213	(1,5,6,2,4,3)	8	2,3,4,5,6,8.		
225	(1,3,4)(5,6)	8	2,3,5,8.		
228	(1,3)(2,4)(5,6)	8	2,3,5,6,8.		
229	(1,3,2,4)(5,6)	4	2,4,6,7.	Alt	
240	(1,3,5)(2,4,6)	2	2,3.	Alt	$\subset S_{168}$
241	(1,3,5,2,4,6)	8	2,3,4,6,8.		
243	(1,2,4,6,3,5)	8	2,3,4,6.		
244	(1,2,4,6)(3,5)	8	2,3,4,5,6,7.	Alt	
257	(1,3,5)(4,6)	4	2,3,4.		
264	(1,4,6,3)(2,5)	2	1,2,6.	Alt	
265	(1,4,6,3,2,5)	8	2,5,6,8.		
267	(1,4,6,2,5,3)	8	2,3,5,6.		
274	(1,3)(2,5)(4,6)	4	1,2,4,8.		
285	(1,2,5)(3,4,6)	8	2,3,4,6,7.	Alt	
308	(1,5,3,2,4,6)	8	2,3,4,5,6,8.		
309	(1,5,3)(2,4,6)	4	1,2,3,6.	Alt	
359	(1,3)(4,6)	1	1,2.	Alt	Aut(A)
407	(1,5,2,4,3,6)	8	2,3,5,6,8.		
423	(1,2,4)(3,6,5)	4	1,3,7.	Alt	
424	(1,2,4,3,6,5)	4	2,3,4,5,6,8.		
427	(1,4)(3,6,5)	8	2,3,4,8.		
428	(1,4,3,6,5)	8	2,3,4,5,6,7.	Alt	
445	(1,4)(3,6)	2	1,4.	Alt	
446	(1,4,3,6)	4	1,2,4.		
459	(1,2,4,5)(3,6)	4	2,3,4,6.	Alt	
460	(1,2,4,5,3,6)	4	2,3,4,5,6,8.		
520	(1,2,6,5)(3,4)	4	1,2,4,7	Alt	
722	(1,3,4,5,6,7,2)	8	2,3,5,7.	Alt	
723	(1,3,4,5,6,7)	8	2,3,4,5,6,8.		
724	(3,4,5,6,7)	4	2,3,5,6,7.	Alt	
732	(1,4,5,6,7,3,2)	8	2,3,4,5,7.	Alt	
738	(4,5,6,7)	1	1,4.		Aut(A)

TABLE 1. (continued)

n.r.	corresp. perm.	T	cycle-lengths	Alt(A)?	Notes
742	(1,3,2)(4,5,6,7)	8	2,3,4,8.		
743	(1,3)(4,5,6,7)	4	1,2,4.	Alt	
744	(1,3,5,6,7,2,4)	4	2,3,6,7.	Alt	
749	(2,4,3,5,6,7)	4	2,3,4,5,6,8.		
750	(1,4,2)(3,5,6,7)	8	3,4,6,8.		
768	(1,4,2,5,6,7,3)	4	2,4,7.	Alt	
786	(1,3,4)(2,5,6,7)	8	2,3,4,6,8.		
787	(1,3,4,2,5,6,7)	4	2,3,6,7.	Alt	
788	(2,5,6,7,3,4)	2	6,8.		
812	(1,5,6,7)(2,4,3)	8	3,4,6,8.		
813	(1,5,6,7,2,4,3)	4	2,4,7.	Alt	
830	(2,4,3)(5,6,7)	4	3,4,7.	Alt	$\subset S_{168}$
834	(1,4,3,2)(5,6,7)	8	3,4,6,8.		
846	(1,4,6,7,3,5,2)	4	7.	Alt	
847	(1,4,6,7,3,5)	8	2,3,4,5,6,8.		
865	(1,4,6,7,3,2,5)	8	2,4,6,7.	Alt	
866	(1,4,6,7)(2,5,3)	8	2,3,4,6.		
908	(1,5,3,2,4,6,7)	8	2,3,4,5,7.	Alt	
962	(1,4,2,5,3,6,7)	4	2,3,4,5,7.	Alt	
975	(1,3,6,7)(2,5,4)	2	2,3,4,5.		
1028	(1,4,3,6,7,5)	8	2,3,4,5,6,8.		
1063	(1,4,5)(3,6,7)	4	1,3.	Alt	
1064	(1,4,5,3,6,7)	2	1,3.		
1124	(1,4,3,2,6,7,5)	4	7.	Alt	
1367	(1,3)(4,5)(6,7)	1	1,2.		
1462	(3,5,7)(4,6)	4	2,3,4.		
1487	(2,5,7)(3,4,6)	2	3,4.	Alt	
1512	(4,6)(5,7)	1	1,2.	Alt	Aut(A)
1628	(1,4,3,6)(5,7)	2	1,2.	Alt	

NOTE: Elementary transcosets embedded in the Alternating Group, Alt(A) are marked "Alt".

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